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ON SOLVING THE DIRICHLET GENERALIZED PROBLEM FOR HARMONIC FUNCTION BY THE METHOD OF FUNDAMENTAL SOLUTIONS

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Abstract. The Dirichlet generalized problem for the Laplace equation in the case of a finite m-connected domain D which lies in the plane $z = x + iy$ is considered. Under the Dirichlet generalized problem is meant the problem when a boundary function has a finite number of first kind points of discontinuity. It is shown that the method of fundamental solutions (MFS) is not suitable for solving of the considered problem. To avoid this situation it is recommended to smooth preliminary the boundary function, i.e., to reduce the generalized problem to an ordinary problem and to solve the latter by the MFS method. Analytic forms of smoothing functions for a finite simply and multiply connected domains are given. Numerical examples are considered to illustrate effectiveness and simplicity of the proposed way.

Keywords and phrases: Dirichlet generalized problem, harmonic function, method of fundamental solutions, break point, smoothing function.

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1. The Dirichlet Generalized Problem for a Harmonic Function

Let D be a finite m-connected domain in the plane $z = x + iy \equiv (x, y)$ with the boundary $S = \bigcup\limits_{k=1}^{m} S_k$, where each S_k is a closed simple contour (Jordan curve) and $S_k \bigcap S_j = \emptyset$ for $k \neq j$. Moreover, we assume that parametric equations of the contours S_k are given and the contours $S_j (j = 1, 2, \dots, m-1)$ lie inside the finite domain which is bounded by the contour S_m.

It is known that the classical statement of the Dirichlet ordinary boundary value problem for the Laplace equation requires continuity of the boundary function. However, in practical problems (for example, during determination of the temperature of the thermal field or of the potential of the electric field and so on) there are cases when the boundary function is piecewise continuous and therefore it is necessary to consider the Dirichlet generalized problem (see [1,2]).

A. *On the boundary S of the domain D a function $g(\tau)$ is given which is continuous everywhere, except a finite number of points $\tau_1, \tau_2, \dots \tau_n$ at which it has break points of the first kind. It is required to find a function $u(z) \equiv u(x, y) \in C^2(D) \bigcap C(\overline{D} \setminus \{\tau_1, \tau_2, \cdots, \tau_n\})$ satisfying the conditions*

$$\Delta u(z) = 0, \qquad z \in D, \tag{1.1}$$

$$u(\tau) = g(\tau), \qquad \tau \in S, \quad \tau \neq \tau_k \; (k = 1, 2, \dots, n), \tag{1.2}$$

$$|u(z)| < M, \qquad z \in \overline{D}, \tag{1.3}$$

where Δ is the Laplace operator and M is a real constant.

It is known [1,2] that Problem (1.1)-(1.3) is correct, i.e., the solution exists, is unique, depends continuously on the data, and for the generalized solution $u(z)$ the generalized extremum principle is valid:

$$\min_{z\in S} u(z) < \underset{z\in D}{u(z)} < \max_{z\in S} u(z), \tag{1.4}$$

where for $z \in S$ it is assumed that $z \neq \tau_k$ $(k = \overline{1,n})$.

Note that the additional requirement (1.3) of boundedness concerns actually only the neighborhoods of break points of the function $g(\tau)$. In particular, if $g^-(\tau_k)$ and $g^+(\tau_k)$ are the limit values of the boundary function $g(\tau)$, when τ tends to the point τ_k along S, respectively, in the positive and negative directions, then the following theorem explains the behavior of the generalized solution in the neighborhood of the point τ_k (see [1,3]).

Theorem 1. *The limit values of the solution $u(z)$ of the Dirichlet generalized problem, when the point $z \in D$ approaches the point τ_k lie between $g^-(\tau_k)$ and $g^+(\tau_k)$.*

It should be noted that condition (1.3) plays an important role in the extremum principle (1.4) and, consequently, in the theorem on uniqueness of the solution to Problem A (see [1]). Evidently, if the function $g(\tau)$ is continuous on S, then the Dirichlet generalized problem coincides with the ordinary problem.

Remark 1. If the domain D is the interior of the circle $S: \ x = a\cos t,\ y = a\sin t$ $(0 \le t \le 2\pi)$, then the solution of the Problem A is represented by Poisson's integral [1,2]:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} g(ae^{it})\frac{a^2 - r^2}{r^2 - 2ar\cos(t-\varphi) + a^2}dt \tag{1.5}$$

where $r < a$ and $z = re^{i\varphi}$ $(0 \le \varphi \le 2\pi)$. When $r = a$ representations (1.5) lose meaning. However, it is proved [1,2] that

$$\lim_{z\to\tau} u(z) = g(\tau), \quad \tau = ae^{it},\ \tau \neq \tau_k,\ z \in D.$$

Remark 2. On the basis of the formula (1.5) the Problem A for simply connected domains can be solved by the method of conformal mapping [1]. In particular, for this it is necessary to know the function $z = \omega(\zeta)$ which conformally maps the unit disk $G(|\zeta| < 1)$ onto a simply connected domain D, and for calculation of the solution to the Problem A at an arbitrary point of the initial domain D (also for determination the pre-images t_k of the points $\tau_k (k = 1, 2, \dots, n)$ during conformal mapping $z = \omega(\zeta)$) it is necessary to know the function $\zeta = f(z)$ which is inverse to the function $z = \omega(\zeta)$. But an exact or approximate construction of noted functions for a given simply connected domain D (except a rather limited family of domains) are very difficult mathematical problems.

2. A note on Solving of the Dirichlet Generalized Problem by the MFS

In general, it is known [4,5] that the methods used for approximate solving of ordinary boundary problems are less suitable (or not suitable at all) for solving problems with boundary singularities. Therefore researchers try to conduct preliminary improvement of the posed boundary problem. More precisely, they try to reduce, if possible, the posed problem by smoothing a boundary function to solving the ordinary problem. For example, the question about application of the MFS to harmonic and biharmonic problems with certain singularities is considered in [6,7,8,9].

Similar case takes place in solving the Dirichlet generalized boundary problem by the MFS. In particular, the convergence is very slow and the accuracy is very low in the neighbourhood of singularity of the boundary function. In general, the MFS may be used for solving both ordinary and generalized problems (see [10,11,12]). In both cases for the finite domain D, the solution is approximated by

$$u_N(z) = \sum_{k=1}^{N} a_k \ln|z - \widetilde{z}_k|, \qquad z \in \overline{D}, \tag{2.1}$$

where the points (singularities) $\widetilde{z}_k$ $(k = 1, 2, \cdots, N)$ lie on the auxiliary contour $\widetilde{S}$. In the approximation (2.1), the number N and the locations of the points $\widetilde{z}_k$ and the coefficients a_k are determined so that $u_N(z)$ satisfies the boundary conditions as well as possible.

Concerning the rate of the convergence and accuracy in the neighbourood of singularity of the boundary function , the noted fact was expected. Indeed, the fundamental solutions (functions) which are participated in (2.1) have a high degree of smoothness on the contour S, therefore, such smooth functions are less suitable for approximation of discontinuous functions. Taking into account the fact that for a very big N computation becomes complicated, the above noted facts makes the MFS less suitable (or not suitable at all) for approximate solving of Problem A.

It is evident that in order to avoid this situation, we should remove the reason of slow convergence of a approximate process. For this, a reduction of Problem A to ordinary problem is necessary.

3. A Method of Reduction of the Dirichlet Generalized Problem to an Ordinary Problem

For reduction of Problem A it is sufficient to have a function $u_0(z)$ which would be a solution of equation (1.1), bounded in $\overline{D}$, continuous in $\overline{D}$ everywhere, except the points $\tau = \tau_k$, and would have the same jumps at the points τ_k, as $g(\tau)$ has. Indeed, if such a function is constructed, then by introduction of a new unknown function

$$v(z) = u(z) - u_0(z), \tag{3.1}$$

for its determination we have already a Dirichlet ordinary problem.

B.

$$\Delta v(z) = 0, \quad z \in D, \tag{3.2}$$

$$v(\tau) = f(\tau), \quad \tau \in S, \tag{3.3}$$

where $f(\tau)$ is a continuous function on the contour S (since the function $f(\tau)$ has removable break points at τ_k, i.e.,$f(\tau_k) = f^-(\tau_k) = f^+(\tau_k)$).

After smoothing of the boundary function $g(\tau)$, the Problem B is solved by MFS, i.e., the approximate solution of the Problem B is sought in the form

$$v_N(z) = \sum_{k=1}^{N} a_k \ln|z - \widetilde{z}_k|, \quad z \in \overline{D}, \tag{3.4}$$

where the points $\widetilde{z}_k (k = 1, 2, \cdots, N)$ are situated "uniformly" on the auxiliary contour $\widetilde{S}$ ($\widetilde{S} \subset R^2 \backslash \overline{D}$) [3,4,5]. As for the coefficients a_k, they can be found from the system:

$$\sum_{k=1}^{N} a_k ln|z_j - \widetilde{z}_k| = f(z_j), \quad (j = 1, 2, \cdots, N), \qquad (3.5),$$

where the collocation points z_j are situated "uniformly" on the contour S (see [6,7,8]. On the basis of (3.1) approximate solution of the Problem A will be

$$u_N(z) = v_N(z) + u_0(z), \quad z \in \overline{D}, \quad z \neq \tau_k. \tag{3.6}$$

For simplicity of presentation the case, when D is simply connected domain is considered separately. In this case in the role of $u_0(z)$ the function (see [1])

$$u_0(z) = \sum_{k=1}^{n} \frac{h_k}{\delta_k} arg(z - \tau_k) \tag{3.7}$$

can be taken, where h_k and δ_k are the jumps of the functions $g(\tau)$ and $arg(\tau - \tau_k)$ at the point τ_k, along S, respectively; In particular

$$h_k = g^+(\tau_k) - g^-(\tau_k), \quad \delta_k = \varphi_k^+ - \varphi_k^-,$$

where $g^-(\tau_k)$, $g^+(\tau_k)$ and φ_k^-, φ_k^+ are the limit values of $g(\tau)$ and $arg(\tau - \tau_k)$ when τ tends to the point τ_k along S, respectively, in the positive and negative directions (by the positive direction the movement along the boundary in the counter-clockwise direction is meant); if τ_k is not an angular point, then $\delta_k = -\pi$; arg denotes the properly chosen branch of the argument; It is evident that a range of the function $arg(z - \tau_k)$ in D depends on a location of the point τ_k on the contour S.

Let D be a finite m-connected domain with the boundary $S = \bigcup\limits_{k=1}^{m} S_k$, where each S_k is a closed simple contour. In this case we can take (see [3])

$$\sum_{i=1}^{l} \sum_{k=1}^{k_i} u_{i,k}(z) \tag{3.8}$$

in the role of the function $u_0(z)$ with

$$u_{i,k}(z) = \begin{cases} \dfrac{h_{i,k}}{\delta_{i,k}} arg\left(\overline{\dfrac{z-\tau_{i,k}}{(z-z_{i,0})(z_{i,0}-\tau_{i,k})}}\right) & for\ \Gamma_i \neq S_m, \\ \dfrac{h_{m,k}}{\delta_{m,k}} arg(z-\tau_{m,k}) & for\ \Gamma_i = S_m, \end{cases} \tag{3.9}$$

where by $\Gamma_i(1 \leq i \leq l)$ those of the contours S_k $(k = 1, 2, \cdots, m)$ are denoted on which the break points lie, $l(1 \leq l \leq m)$ is the number of these contours; τ_{ik} $(k = 1, 2, \cdots, k_i)$ are the break points on the contour Γ_i; k_i is the number of break points on the contour Γ_i(it is evident that $1 \leq k_i \leq n$ and $(k_1 + k_2 + \cdots + k_l = n)$; $z_{i,0}$ is a "center" of the finite domain B_i with the boundary Γ_i $(z_{i,0} \in B_i, i \neq m)$, while

$$\delta_{i,k} = \varphi^+_{i,k} - \varphi^-_{i,k},$$

$$\varphi^+_{i,k} = \lim_{\tau \to \tau_{i,k}+} arg\left(\overline{\frac{\tau-\tau_{i,k}}{(\tau-z_{i,0})(z_{i,0}-\tau_{i,k})}}\right),$$

$$\varphi^-_{i,k} = \lim_{\tau \to \tau_{i,k}-} arg\left(\overline{\frac{\tau-\tau_{i,k}}{\tau-z_{i,0})(z_{i,0}-\tau_{i,k})}}\right), \quad \tau \in \Gamma_i,\ \Gamma_i \neq S_m;$$

$$\delta_{m,k} = \varphi^+_{m,k} - \varphi^-_{m,k},$$

$$\varphi^+_{m,k} = \lim_{\tau \to \tau_{m,k}+} arg(\tau-\tau_{m,k}),$$

$$\varphi^-_{m,k} = \lim_{\tau \to \tau_{m,k}-} arg(\tau-\tau_{m,k}), \quad \tau \in S_m.$$

In the formulas the sign "—-" denotes complex conjugate.

4. Examples of Solving Generalized Problems

In this section on the basis of considered scheme the results of approximate solving of generalized problem for simply and doubly connected domains are given. In all examples considered below the coefficients a_k of expansion (3.4) are found from system (3.5).

In the Tables, N is the number of auxiliary and collocation points on the contours $\widetilde{S}$ and S, respectively; ε is an *a posteriori* error estimate of the solution of the problem (3,2),(3,3):

$$\varepsilon = max|f(z_i) - v_N(z_i)|,$$

where $f(z_i) = g(z_i) - u_0(z_i)(z_i \neq \tau_k)$; The points $z_i(i = 1, 2, \cdots, M)$ are situated "uniformly" on the contour S. If $z_i = \tau_k$, then $f(z_i) = f^+(\tau_k) \equiv f^-(\tau_k)$. In numerical experiments $M = 10000$ was taken.

Example 1. Let the domain D be the interior of the circle $S : x = 2\cos t,\ y = 2\sin t$ $(0 \leq t \leq 2\pi)$, in the role of optimal axiliary contour $\widetilde{S}$ (in the sense of accuracy of approximate solution) for the given N the circle $\widetilde{S} : x = 2.02\cos t, \quad y = 2.02\sin t$

$(0 \le t \le 2\pi)$ is taken in solving Problem B by MFS. We took a function with one break point $\tau_1 = (2, 0)$ as function $g(\tau)$. In particular, we took the function

$$g(\tau) = \begin{cases} x^2 - y^2 - 4, & (x, y) \equiv \tau \in \tau_1\tau_2, \ \ (y \ge 0), \\ \frac{1}{\ln 3} \ln|\tau - z_0|, & \tau \in \tau_2\tau_1, \ \ y \le 0, \end{cases}$$

where $\tau_1\tau_2, \tau_2\tau_1$ are open arcs of the contour S, and $\tau_2 = (-2, 0)$ and $z_0 = (-1, 0)$.

For the considered case: $\frac{\pi}{2} \le arg(z - \tau_1) \le \frac{3\pi}{2}$, $z \in \overline{D}$, $z \ne \tau_1$; $g^+(\tau_1) = 0$, $g^-(\tau_1 = 1$, $\varphi_1^+ = \frac{\pi}{2}$, $\varphi_1^- = \frac{3\pi}{2}$, therefore $h_1 = -1$ and $\delta_1 = -\pi$ (see (3.7)).

Auxiliary points, collocation points and points for calculation the *a posteriori* estimate ε in this and other examples are situated uniformly with respect to the parameter t on the contours $\widetilde{S}$ and S, respectively. In the Table 1, $u_N(z_k))$ is the value of approximate solution to the Problem A at the point $z_k \in D$ which is calculated with (3.6); $u(z_k)$ is value of exact solution to the Problem A at point $z_k \in D$ which is calculated by Poisson's integral (1.5).

Table 1

$N = 1500$; $\varepsilon = 0.610^{-5}$			
k	z_k	$u_N(z_k)$	$u(z_k)$
1	$(0, 0)$	-1.68453512403465	-1.68453512318792
2	$(1, 1)$	-3.42897119266898	-3.42897119234831
3	$(1.99999, 0)$	0.49997924070930	0.49997903692072

Example 2. The domain D is the interior of the ellipse $S : x = 2\cos t$, $y = \sin t (0 \le t \le 2\pi)$. In the role of $g(\tau)$ we took a function with four break points: $\tau_1 = (2, 0)$, $\tau_2 = (0, 1)$, $\tau_3 = (-2, 0)$, $\tau_4 = (0, -1)$. In particular, we took the function

$$g(\tau) = \begin{cases} x + y, & (x, y) \equiv \tau \in \tau_1\tau_2 \\ \ln(x^2 + y^2), & \tau \in \tau_2\tau_3 \\ x + y, & \tau \in \tau_3\tau_4 \\ \ln(x^2 + y^2), & \tau \in \tau_4\tau_1, \end{cases}$$

where $\tau_1\tau_2$, $\tau_2\tau_3$, $\tau_3\tau_4$, $\tau_4\tau_1$ are open arcs of the contour S. In the considered case: $g^+(\tau_1) = 2$, $g^-(\tau_1) = 2\ln 2$; $g^+(\tau_2) = 0$, $g^-(\tau_2) = 1$; $g^+(\tau_3) = -2$, $g^-(\tau_3) = 2\ln 2$; $g^+(\tau_4) = 0$, $g^-(\tau_4) = -1$, and $h_1 = 2 - 2\ln 2$, $h_2 = -1$, $h_3 = -2 - 2\ln 2$, $h_4 = 1$, $\delta_k = -\pi$ $(k = 1, 2, 3, 4)$.

For illustration, the form of graph of the function $g(\tau)$ is given in the Figure a), and the form of graph of the function $f(\tau)$ is given in the Figure b). In solving the Problem B by MFS we took the ellipse $\widetilde{S}$ in role of the auxiliary contour $\widetilde{S}$: $x = 2.005\cos t$, $y = 1.005\sin t$ $(0 \le t \le 2\pi)$

In the Table 2 the values of approximate solution to the Problem A calculated by (3.6) on the various points $z_k \in D$ are given.

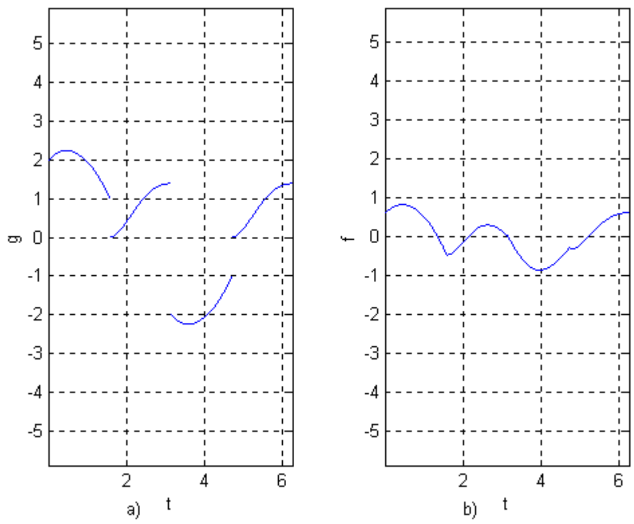

Table 2

$N = 2000 \quad \varepsilon = 0.510^{-3}$		
k	z_k	$u_N(z_k)$
1	(0.0)	0.192275110
2	$(0.5, 0.5)$	1.04613621615
3	$(-0.5, 0.5)$	0.1775304638
4	$(-0.5, -0.5)$	-0.792471601
5	$(0.5, -0.5)$	0.3552674521
6	$(1.99999, 0)$	1.693234555
7	$(0, 0.99999)$	0.50001643577
8	$(-1.99999, 0)$	-0.306969584
9	$(0, -0.99999)$	-0.5000780554

Example 3. The domain D is a finite doubly connected domain with the boundary $S = S_1 \cup S_2$ where the contour S_1 $(S_1 \equiv \Gamma_1)$ is the ellipse $S_1 : x = -3 + 2\cos t,\ y = \sin t$ $(0 \le t \le 2\pi)$ and the contour S_2 $(S_2 \equiv \Gamma_2$ is circle $S_2 :\ x = 10\cos t,\ y = 10\sin t$ $(0 \le t \le 2\pi)$.

In the role a boundary function $g(\tau)$ we took the function

$$g(\tau) = \begin{cases} g_1(\tau), & \tau \in S_1, \\ \\ g_2(\tau), & \tau \in S_2 \end{cases} \tag{4.1}$$

In (4.1) the functions $g_1(\tau)$ and $g_2(\tau)$ have the form

$$g_1(\tau) = \begin{cases} 1, & \tau \in \tau_{1,1}\tau_{1,2}\ , \\ \\ 2, & \tau \in \tau_{1,2}\tau_{1,3}\ , \\ \\ 3, & \tau \in \tau_{1,3}\tau_{1,4}\ , \\ \\ 4, & \tau \in \tau_{1,4}\tau_{1,1}\ ; \end{cases}$$

$$g_2(\tau) = \begin{cases} 1, & \tau \in \tau_{2,1}\tau_{2,2}\ , \\ 3, & \tau \in \tau_{2,2}\tau_{2,3}\ , \\ 5, & \tau \in \tau_{2,3}\tau_{2,4}\ , \\ 7, & \tau \in \tau_{2,4}\tau_{2,1}, \end{cases}$$

on the contours S_1 and S_2, respectively.

It is evident that the jumps of the function $g(\tau)$ at the break points: $\tau_{1,1} = (-1,0), \tau_{1,2} = (-3,1), \tau_{1,3} = (-5,0), \tau_{1,4} = (-3,-1), \tau_{2,1} = (10,0), \tau_{2,2} = (0,10), \tau_{2,3} = (-10,0), \tau_{2,4} = (0,-10)$, respectively are equal to: $h_{1,1} = -3, h_{1,2} = 1, h_{1,3} = 1, h_{1,4} = 1, h_{2,1} = -6, h_{2,2} = 2, h_{2,3} = 2, h_{2,4} = 2$;

On the basis of the Section 3 for smoothing of the boundary function (4.1) we used the functions (3.8) and (3.9), in which $l = 2, m = 2, \delta_{ik} = -\pi$ $(i = 1,2;\ k = 1,2,3,4;\ k_i = 4)$, and $z_{1,0} = (-3,0)$.

In solving of the Problem B by MFS in the role of contours $\widetilde{S}_1$ and $\widetilde{S}_2$ we took the contours $\widetilde{S}_1 :\ x = -3 + 2.01\cos t,\ y = 1.01\sin t$ and $\widetilde{S}_2 :\ x = 10.01\cos t,\ y = 10.01\sin t,\ (0 \le t \le 2\pi)$.

In numerical experiment the number of collocation (auxiliary) points on the contours S_1 and S_2 ($\widetilde{S}_1$ and $\widetilde{S}_2$) were equal $N_1 = 800$ and $N_2 = 800$, i.e., $N = 1600$. Analogously $M = M_1 + M_2$, where $M_1 = M_2 = 5000$.

In the Table 3 the values of approximate solution $u_N(z)$ of the Problem A calculated by (3.6) on the various points $z_k \in D$ are given.

Table 3

$N = 1600;$		$\varepsilon = 10^{-3}$
k	z_k	$u_N(z_k)$
1	$(9.999, 0)$	4.0000619
2	$(0, 9.999)$	1.999971
3	$(-9.999, 0)$	3.9996653
4	$(0, -9.999)$	5.99981577
5	$(-0.999, 0)$	2.50078230
6	$(-3, 1.001)$	1.50039327
7	$(-5.001, 0)$	2.500895444
8	$(-3, -1.001)$	3.500318976
9	$(5, 0)$	3.69552249

5. Concluding Remarks

From Tables 1,2,3 it seems that for the approximate solution $u_N(z)$ of the Problem A at the considered points of the domain D, the conditions of the generalized extremum principle and Theorem 1 are valid.

It is known [7] that for fixed N there exist an optimal auxiliary contour $\widetilde{S}$ (or optimal location of auxiliary points) in sense of accuracy of approximate solution of

Problems type B. In [7], it is shown that: 1) if for given N, corresponding optimal contour $\widetilde{S}$ is moving off contour S then the conditionality of the matrix of system (3.5) is deteriorated and on the contrary, by moving $\widetilde{S}$ to the S, improving. On the basis of noted, for construction with the move high accuracy of the approximate solution $u_N(z)$, in the considered examples, it is sufficient simultaneously to increase of N and to approach of $\widetilde{S}$ to S.

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