Proceedings of A. Razmadze
Mathematical Institute
Vol. **160** (2012), 71–89

A METHOD OF CONFORMAL MAPPING FOR SOLVING THE GENERALIZED DIRICHLET PROBLEM OF LAPLACE'S EQUATION

M. KUBLASHVILI, Z. SANIKIDZE AND M. ZAKRADZE

Abstract. In this paper we investigate the question how the method of conformal mapping (MCM) can be applied for approximate solving of the generalized Dirichlet boundary problem for harmonic function. Under the generalized problem is meant the case when a boundary function has a finite number of first kind break points. The problem is considered for finite and infinite simply connected domains. It is shown that the method of fundamental solutions (MFS) is ineffective for solving of the considered problem from the point of view of the accuracy. We propose an efficient algorithm for approximate solving of the generalized problem, which is based on the MCM. Examples of application of the proposed algorithm and the results of numerical experiments are given.

რეზიუმე. შესწავლილია საკითხი, თუ როგორ შეიძლება კონფორმულ გადასახვათა მეთოდი გამოყენებული იქნას ჰარმონიული ფუნქციისათვის დირიხლეს განზოგადებული სასაზღვრო ამოცანის მიახლოებით ამოხსნისათვის. განზოგადებული ამოცანის ქვეშ იგულისხმება შემთხვევა, როცა სასაზღვრო ფუნქციას აქვს პირველი გვარის წყვეტის წერტილების სასრული რაოდენობა. ამოცანა განხილულია სასრული და უსასრულო ცალადბმული არეებისათვის. ნაჩვენებია, რომ სიზუსტის თვალსაზრისით ფუნდამენტურ ამოხსნათა მეთოდი არ არის ეფექტური განხილული ამოცანის მიახლოებით ამოხსნისთვის. შემოთავაზებულია ეფექტური ალგორითმი განზოგადებული ამოცანის მიახლოებით ამოხსნისათვის, რომელიც დაფუძნებულია კონფორმულ გადასახვათა მეთოდზე. მოცემულია აღნიშნული მეთოდის გამოყენების მაგალითები და რიცხვითი ექსპერიმენტების შედეგები.

1. Introduction

Let a domain D in the plane $z = x + iy \equiv (x, y)$ be bounded by a closed piecewise smooth contour S without multiple points (i.e., S is a simple contour). Moreover, we assume that its parametric equation is given.

2010 *Mathematics Subject Classification.* 30C30, 65C05, 65N38, 65N99.

Key words and phrases. Generalized Dirichlet problem, Laplace's equation, method of conformal mapping, break point.

It is known that in the Dirichlet ordinary boundary value problem for the Laplace equation requires the continuity of the boundary function. However, in practical problems (e.g., at determining the temperature of the thermal field or of the potential of the electric field and so on) there are cases when the boundary function is piecewise continuous and therefore it is necessary to consider the generalized Dirichlet problem (see [1,2]).

A. *On the boundary S of the domain D a function $g(\tau)$ is given which is continuous everywhere, except a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ at which it has discontinuities of the first kind. It is required to find a function $u(z) \equiv u(x,y) \in C^2(D) \bigcap C(\overline{D} \setminus \{\tau_1, \tau_2, \dots, \tau_n\})$ satisfying the conditions*

$$\Delta u(z) = 0, \quad z \in D, \tag{1.1}$$

$$u(\tau) = g(\tau), \quad \tau \in S, \quad \tau \neq \tau_k \ (k = 1, 2, \dots, n), \tag{1.2}$$

$$|u(z)| < M, \quad z \in \overline{D}, \tag{1.3}$$

where Δ is the Laplace operator and M is a real constant.

In what follows, we assume that the points τ_k are situated on the contour S preserving the order of succession under the positive circuit of S (The movement along the boundary in the counter- clockwise direction is meant by the positive direction).

Note that the additional requirement of boundedness, when the domain D is finite, concerns actually only the neighborhoods of break points of the function $g(\tau)$. If the domain D is infinite, then condition (1.3) (except the above-mentioned) means that (see [3])

$$\lim u(z) = c, \quad \text{for } z \to \infty,$$

where c is a real constant and $|c| < \infty$.

It is known [1,2] that Problem (1.1)–(1.3) is correct, i. e., the solution exists, is unique, depends continuously on the data, and for the generalized solution $u(z)$ the generalized extremum principle is valid:

$$\min_{z \in S} u(z) < \underset{z \in D}{u(z)} < \max_{z \in S} u(z),$$

where for $z \in S$ it is assumed that $z \neq \tau_k$ $(k = \overline{1,n})$.

If $g^-(\tau_k)$ and $g^+(\tau_k)$ are the limit values of the boundary function $g(\tau)$, when τ tends to the point τ_k along S, respectively, in the positive and negative directions, then the following theorem explains the behavior of the generalized solution in the neighborhood of the point τ_k (see [1]).

Theorem 1. *The limit values of the solution $u(z)$ of the generalized Dirichlet problem, when the point $z \in D$ approaches the point τ_k lie between $g^-(\tau_k)$ and $g^+(\tau_k)$.*

Evidently, if the function $g(\tau)$ is continuous on S, then the generalized Dirichlet problem coincides with the ordinary problem.

Such practical problems (physical processes) the investigation of which are reduced to solving the generalized Dirichlet problem for Laplace's (or Poisson's) equation are considered in [4].

If the domain D is the interior (or the exterior) of the circle $S: \ x = a\cos\varphi, \ y = a\sin\varphi \ (0 \le \varphi \le 2\pi)$, then the solution of the Problem A is represented by Piosson's integral [1,2]. In particular, the solution of the interior problem has the form

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} g(ae^{i\varphi})\frac{a^2 - r^2}{r^2 - 2ar\cos(\varphi - \theta) + a^2}d\varphi \quad \text{for} \quad r < a, \tag{1.4}$$

and the solution of exterior problem has the form

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} g(ae^{i\varphi})\frac{r^2 - a^2}{r^2 - 2ar\cos(\varphi - \theta) + a^2}d\varphi \quad \text{for} \quad r > a, \tag{1.5}$$

where $z = re^{i\theta}$ $(0 \le \theta \le 2\pi)$. When $r = a$ representations (1.4) and (1.5) lose the sense. However, it is proved [1,2] that

$$\lim_{z\to\tau} u(z) = g(\tau), \quad \tau = ae^{i\varphi}, \quad \tau \ne \tau_k, \quad z \in D.$$

2. A Note on Solving of the Generalized Dirichlet Problem by the MFS

2.1. **MFS formulation for harmonic problems.** The basic ideas for the formulation of the MFS were first proposed by Kupradze and Aleksidze [5]. Consider the Dirichlet problem for Laplace's equation

$$\Delta v(z) = 0, \quad z \in D, \tag{2.1}$$

$$v(z) = b(z), \quad z \in S, \tag{2.2}$$

where D is a bounded domain with its boundary S, and $b(z) = b(x, y)$ is a continuous function defined on S.

The MFS approximates the solution $v(z)$ by

$$v_N(z) = \sum_{k=1}^{N} a_k \ln|z - \widetilde{z}_k|, \quad z \in \overline{D}, \tag{2.3}$$

namely, a linear combination of fundamental solutions of Laplace's equation, where the points (singularities) $\widetilde{z}_k$ $(k = 1, 2, \dots, N)$ are situated uniformly (in a sense) outside the domain D so-called on the closed auxiliary contour $\widetilde{S}$. The contour $\widetilde{S}$ contains the domain $\overline{D} = D + S$ and $\min\rho(S, \widetilde{S}) > 0$, where ρ is the distance between the curves S and $\widetilde{S}$. In approximation (2.3), the number N and the locations of the points $\widetilde{z}_k$ and the coefficients $a_k (k = 1, 2, \dots, N)$ are determined so that $v_N(z)$ satisfies the boundary condition as well as possible. In particular, the coefficients a_k are determined to satisfy the boundary condition (2.2) at the collocation points $z_j (j = 1, 2, \dots, N)$, which are situated uniformly (by the same law as the points $\widetilde{z}_k$) on the boundary S. That is to say, they are solution of a system of N linear equations

$$\sum_{k=1}^{N} a_k \ln |z_j - \widetilde{z}_k| = b(z_j), \quad (j = 1, 2, \dots, N),$$

which is called the collocation condition. Once a_k $(k = 1, 2, \dots, N)$ are determined, $v(z)$ can be approximated by $v_N(z)$ at any point $z \in \overline{D}$.

The approximation $v_N(z)$ exactly satisfies the Laplace equation (2.1). Consequently, if the domain D is bounded, by the maximum principle for harmonic functions, computational error of a solution of the problem (2.1), (2.2) can be estimated as

$$\varepsilon = \max_{z \in S} \big| v_N(z) - b(z) \big| \approx \max \big| v_N(z_j) - b(z_j) \big|,$$

where the points z_j $(j = 1, 2, \dots, M)$ are situated uniformly on the contour S and $M >> N$.

It should be noted that the application and investigation of the MFS have been carried out in the course of 50 years. In various scientific centers have extensively used the MFS to solve complicated problems such as conformal mapping problems, linear and nonlinear problems of the potential theory, boundary value problems with a free boundary, biharmonic problems, problems of elastostatics and elastodynamics, direct and inverse problems of geophysics and so on.

A special mention should be made of a wide application of the MFS for investigation of a number of boundary value problems. For instance, the MFS plays the main role in proving the existance theorems in mixed dynamic problems as well as in dynamic of the moment theory of elasticity and thermoelasticity (see [6]).

From the mathematical standpoint the MFS is studied in a number of works, of which we should mention Christiansen [7], Mathon and Johnston [8], Fairweather and Jonston [9], Katsurada and Okamoto [10], Katsurada [11, 12], Kitagawa [13], Fairweather and Karageorghis [14], Smyrlis and Karageorghis [15].

2.2. **On application of the MFS for generalized problem.** In general, it is known [4, 16] that the methods used for approximate solving the ordinary boundary problems are less suitable (or not suitable at all) for solving problems with singularities. In particular, the convergence is very slow and, consequently, the accuracy is very low in the neighborhood of singularity of the boundary function. Similar case takes place in solving the generalized Dirichlet boundary problem by the MFS. Therefore researchers try to conduct preliminary improvement of the posed boundary problem. More precisely, they try to reduce, if possible, the posed problem by smoothing a boundary function to solving the ordinary problem (see e.g., [4, 16, 17, 18]. For example, the question about application of the MFS to harmonic and biharmonic problems with certain singularities is considered in [19, 20, 21]. In these papers is noted the MFS for solving harmonic and biharmonic problems with boundary singularities is ineffective from the point of view of the accuracy in the neighborhood of boundary singularities. Therefore, authors for solving the considered problems have used so-called modified versions of the MFS, which are based on the direct subtraction of the leading terms of the singular local solution (which must be determined) from the original mathematical problem. Because of difficulty numerical realization they have considered cases only with the one boundary singularity.

In general, the MFS may be used for solving both ordinary problem and generalized problem (see [5, 17]). Concerning the rate of the convergence and accuracy in the neighborhood of singularity of the boundary function, the noted fact was expected. Indeed, the fundamental solutions (functions) which participate in (2.3) have a high degree of smoothness on the contour S, therefore, such smooth functions are less suitable for approximation of discontinuous functions. Taking into account the fact that for very big N computation becomes complicated, then the above noted facts make the MFS less suitable (or not suitable at all) for approximate solving the Problem A. An analogous circumstance takes place when D is infinite domain. Thus, in the case of generalized problem the MFS is ineffective from the point of view of the accuracy.

3. On Application the MCM for Solving the Generalized Dirichlet Problem

Let in the plane z be given a finite (or infinite) simply connected domain D with a piecewise smooth boundary S, i.e., $\big(S = \bigcup\limits_{j=1}^{l} S_j\big)$. We assume that parametric equations of the lines $S_j : z = z^j(\varphi) \equiv x^j(\varphi) + iy^j(\varphi), \alpha_j \leq \varphi \leq \ \beta_j$ is given.

As it often happens, the problem can be solved in a relatively simple way under more complicated boundary conditions for canonical domains such as a disk, a circular ring, a square and so on (see, e.g., (1.4), (1.5)). Hence, there are attempts to transfer the boundary problem posed for the initial (basic) domain D to the canonical domain G with boundary γ. Obviously in this case generally the following is being changed: 1) The given differential equation; 2) The domain in which the unknown function is sought; 3) The boundary conditions.

As early as the middle of the 19-th century conformal mappings are widely used for transference of a number of plane problems of mathematical physics to canonical domains. To implement the transference a analytic function $z = \omega(\zeta)$ conformally mapping the domain G in the plane $\zeta = \xi + i\eta$ onto the domain D is applied (see e.g., [1, 16, 22, 23]).

The range of problems solvable by the MCM is very wide. In particular, the method has been applied successfully in problems of hydro and aerodynamics, elasticity, filtration etc.

Thus many boundary problems can be reduced to a problem of finding the function $z = \omega(\zeta)$. Note that the solution of boundary problems can easily be constructed when the function $z = \omega(\zeta)$ is either a rational or polynomial. The mentioned circumstance was organically connected with the development of methods for constructing conformally mapping functions.

Since conformally mapping functions $z = \omega(\zeta)$ can be written in explicit form only for a rather narrow family of domains, one has usually to resort to approximate methods of constructing mapping functions appeared (see e.g., [16, 23, 24, 25, 26, 27]).

It should be noted that in solving boundary problems by the MCM the following circumstance takes place [28]. The function $z = \widetilde{\omega}(\zeta)$ which is constructed approximately, maps conformally the canonical domain G onto the domain $\widetilde{D}$ which is close to D, and thus, practically, the problem stated for the domain D with boundary S is being solved for the domain $\widetilde{D}$ with the simple boundary $\widetilde{S}$. Here we mean that the conditions which ensure the existence and uniqueness of a solution to a mathematical problem are fulfilled for the domain $\widetilde{D}$.

The possibility of such approuch is due to the following facts. When passing from a practical problem to a mathematical model, the idealization of both the physical properties of the medium and the contour S takes place. Since the real boundary does not coincide with the ideal boundary S, the contour S has a tolerance field in which it can vary almost arbitrarily (without change of type). Physically this means that a small change of data induces a small change of corollary, and mathematically this means that a solution depends continuously on the data. Therefore, during solving correct problems by the MCM, we have to find a function $z = \widetilde{\omega}(\zeta)$ such that the deviation of a simple contour $\widetilde{S}$ from the given boundary S be within admissible limits. It is evident, that if $\widetilde{S} \to S$, then $\widetilde{u}(x, y) \to u(x, y)$, where $u(x, y)$ is a solution of the initial problem, and $\widetilde{u}(x, y)$ is a solution of the problem for the domain $\widetilde{D}$ with boundary $\widetilde{S}$.

It is known [16, 23] that if the transfer on the domain G is done by the analytic function $z = \omega(\zeta)$ which conformally maps the domain G onto D, then in the case of the Problem A changes only the domain and the boundary conditions. In particular, under the conformal mapping $z = \omega(\zeta)$ the piecewise continuous boundary function $g(\tau)$ $(\tau \in S, \tau \neq \tau_k)$ goes into the piecewise continuous function $g^*(t) = g(\omega(t))$ $t \in \gamma$, which has the same jumps at the point $t_k \in \gamma$ $(\tau_k = \omega(t_k))$, as $g(\tau)$ has. The function $u^*(\zeta) = u(\omega(\zeta))$ is harmonic in the domain G, bounded in $\overline{G}$, continuous in $\overline{G}$ everywhere, except the points $t = t_k$ and $u^*(\zeta) \to g^*(t)$ for $\zeta \to t, t \neq t_k, \zeta \in G, t \in \gamma$.

Thus, in particular, if the domain G is the unit disk $G(|\zeta| < 1)$,then after transference to G we obtain again the Dirichlet generalized boundary Problem B for the Laplace's equation with changed right hand sides, but for the disk G.

B.

$$\Delta u^*(\zeta) = 0, \quad \zeta \in G, \tag{3.1}$$

$$u^*(t) = g^*(t), \quad t \in \gamma, \quad t \neq t_k, \tag{3.2}$$

$$|u^*(\zeta)| < M, \quad \zeta \in \overline{G}, \tag{3.3}$$

where $u^*(\zeta) = u(\omega(\zeta))$, $\zeta \in G$; $g^*(t) = g(\omega(t))$, $t \in \gamma; g^*(t) \in C(\gamma)$ for $t \neq t_k$.

On the basis of formula (1.4) we have

$$u^*(\zeta) = u^*(r, \theta) = \\ = \begin{cases} \dfrac{1}{2\pi} \int\limits_0^{2\pi} g^*(e^{i\varphi}) \dfrac{1 - r^2}{1 - 2r\cos(\varphi - \theta) + r^2}\, d\varphi & \text{for } r < 1, \\ g^*(t),\ t \neq t_k & \text{for } r = 1, \end{cases} \tag{3.4}$$

where $\zeta = re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

Thus, the MCM permits to obtain the representation of the solution of the Problem A for an arbitrary simply connected domain D by Poisson's integral (3.4). However, for this, in general, it is necessary to know the function $\zeta = f(z)$ which is inverse of the function $z = \omega(\zeta)$, i.e., $\zeta = \omega^{-1}(z)$. Indeed, in the first place, in order to calculate integral (3.4) we must know the pre-images t_k of the points τ_k $(k = 1, 2, \dots, n)$ under conformal mapping $z = \omega(\zeta)$ (i.e., $\tau_k = \omega(t_k)$, $\tau_k \in S$, $t_k \in \gamma$). Exact or approximate definition of the points t_k is a difficult problem, and if the function $\zeta = f(z)$ is known, then $t_k = f(\tau_k)$.

Moreover, in general, for calculation of the solution of the Problem A at the arbitrary point of the initial domain D, it is necessary to know the function $\zeta = f(z)$. Indeed, since the functions $\zeta = f(z)$ and $z = \omega(\zeta)$ are mutually inverse, therefore at any point z of the domain D, $u(z) = u^*(\zeta)$, where $\zeta = f(z)$, and the value of the function $u^*(\zeta)$ at the point ζ will be defined by (3.4).

4. The Algorithm of Solving the Generalized Dirichlet Problem

In this section we propose one algorithm for an approximate solving of Problem A. The proposed algorithm is sufficiently simple for numerical realization and it is characterized by accuracy, which is practically sufficient for many problems. Respectively to the general principle which is described in Section 3, the algorithm consists of three stages.

4.1. **An approximate construction of the function $\zeta = f(z)$ for a finite domain.** Let $\zeta = f(z)$ be a function conformally mapping the simply connected finite domain D (bounded by a simple closed contour S) to the unit disk G under normalization conditions

$$f(z_0) = \zeta_0, \quad f(z_1) = \zeta_1, \tag{4.1}$$

where $z_0, z_1, \zeta_0, \zeta_1$ are fixed points and $z_0 \in D, z_1 \in S, \zeta_0 \in G, \zeta_1 \in \gamma$. Without loss of generality we may mean that $\zeta_0 = 0$ and $\zeta_1 = 1$. According to Riemann theorem the function $f(z)$ exists under conditions (4.1) and is determined uniquely. The approximate construction of the function $\zeta = f(z)$ can be realized by using one simple scheme given in [29, 30]. The mentioned scheme is based on the well-known representation of the unknown function [1, 16]

$$f(z) = (z - z_0)e^{u(x,y)+iv(x,y)}, \quad z \in \overline{D},$$

where $u(x, y)$ is a solution of the Dirichlet boundary value problem

$$\Delta u(x, y) = 0, \quad (x, y) \in D, \tag{4.2}$$

$$u(x, y) = -\ln|z - z_0|, \quad z \in S, \tag{4.2$_0$}$$

and $v(x, y)$ is the function harmonically conjugate to $u(x, y)$ and defined to within the constant summand c.

For an approximate solution of problem (4.2), (4.2$_0$) we use the MFS. In that case the differential equation (4.2) is satisfied exactly, while the boundary condition (4.2$_0$) approximately. Thus the error source is only the error of approximation of the boundary function $b(z) = -\ln|z - z_0|$.

Note that practical choice of an auxiliary contour $\widetilde{S}$ depends on a number of factors stipulated by the meaning of the problem and requires certain electoral approach. Numerous experiments have shown that high accuracy of approximation of the boundary function $b(z)$ is in our case reached in such a combination of the contour $\widetilde{S}$ and the point z_0 when $\widetilde{S}$ is similar to S; the contour $\widetilde{S}$ is the boundary of the figure $\widetilde{D}$ similar to figure D and oriented likewise, and the point z_0 is the centroid (in a sense) of plane figures D and $\widetilde{D}$ (see [30, 31]).

Using the MFS and having found $u_N(x,y)$, we can immediately find the corresponding harmonically conjugate to $u_N(x,y)$ function $v_N(x,y)$:

$$v_N(x,y) \equiv \sum_N (z) + c = \sum_{k=1}^{N} a_k \arg(z - \widetilde{z}_k) + c, \tag{4.3}$$

where c is an arbitrary constant. This circumstance essentially simplifies the finding of a conformally mapping function.

It can be easily shown that if the boundary value problem (4.2), (4.2_0) is solved to within ε, i.e., $|u_N(x,y) - u(x,y)| < \varepsilon$, $(x,y) \in S$ then for the absolute value of the function

$$f_N(z) = (z - z_0)e^{u_N(x,y)+iv_N(x,y)}$$

the estimate

$$e^{-\varepsilon} < |f_N(z)| < e^{\varepsilon}, \quad z \in S$$

is valid, and $\lim\limits_{N\to\infty} f_N(z) = f(z)$ (uniformly) for all $z \in \overline{D}$.

If in (4.3) we take

$$c = -\sum_N (z_1) - \arg(z_1 - z_0),$$

then $f_N(z_0) = 0$, $f_N(z_1) = \zeta_1 = 1$ i.e., the normalization conditions (4.1) will be fulfilled.

Remark. When constructing approximately conformally mapping functions, we rely on the following rather useful theorem (Osgood W.F) [1, 22].

Theorem 2. *Let D be a finite or infinite simply connected domain in the plane z, bounded by a simple closed contour S and let $\varphi(z)$ be a function, analytic in D (including the point at infinity, if the domain D is infinite) and continuous in $\overline{D}$. Further, let the point, define by equality $\zeta = \varphi(z)$ describes in the plane ζ (moving always in one and the same direction) some simple closed contour γ^*, when z describes the contour S. Then the relation $\zeta = \varphi(z)$ gives the conformal mapping of the domain D on the domain G^*, enclosed inside γ^* and conversly.*

It should be noted that this theorem is generalized also for the case of multiply-connected domains [22].

Thus, on the basis of the Theorem 2, if $\arg f_N(z)$ increases monotonically in a rigorous sense from 0 to 2π for a single circuit of the curve S (counter-clockwise, starting from the point z_1), then the function $f_N(z)$ provides conformal mapping of the domain D onto the circle G with accuracy $\varepsilon_1 = e^{\varepsilon} - 1$. If the above-mentioned condition is not fulfilled, then we must increase the accuracy of solving the boundary value problem (4.2), (4.2_0).

On the basis of the above-cited algorithm, if we construct the function $\zeta = f_N(z)$ mapping conformally the domain D onto the circle G with accuracy

ε_1, then it becomes clear that the points $t_k^{(N)} = f_N(\tau_k)$ $(k = 1, 2, \dots, n)$ will be located on the contour $\zeta = f_N(z)$, $z \in S$ lying in the ring $e^{-\varepsilon} < \zeta < e^{\varepsilon}$, and these points are in fact the values of the points t_k to within ε_1. However, we can increase the accuracy of determining the points t_k. Indeed, if we carry the points $t_k^{(N)}$ over the circumference γ in the normal γ, then as a result we obtain the points t_k^* which on the basis of the inequality $|t_k^* - t_k| \leq |t_k^{(N)} - t_k|$ (for sufficiently small ε) will be closer to the exact values of the points t_k than the points $t_k^{(N)}$. Hence, since the points t_k^* $(k = 1, 2, \dots, n)$ are the coordinates of the points of intersection of the normal and the curve γ, for an approximate determination of the points t_k we will finally have the expression

$$t_k^* = \frac{f_N(\tau_k)}{|f_N(\tau_k)|} \quad (k = 1, 2, \dots, n). \tag{4.4}$$

It should be noted that if the domain D is infinite then for application of our algorithm it is necessary to transfer the Problem A to the finite simply connected domain D^*. For this we can apply inversion

$$z^* = z_0 + \frac{1}{z - z_0},$$

where z_0 is the point of finite domain E bounded by contour S.

4.2. **An approximate construction of the function $z = \omega(\zeta)$ in a polynomial form.** Below we offer one scheme for approximate construction of the function conformally mapping a unit disk G to the given simply connected domain D (in more detail see [27]). The offered method, realizing the definition of the unknown function in the polynomial form is quite simple for numerical realization and has accuracy which is practically sufficient for many problems.

Note that one of the essential moments in the offered here approach is the fact that by the corresponding scheme on the initial step the function $\zeta = f_N(z)$ is found (see p. 4.1), which realizes approximate conformal mapping of the domain D to the disk G, and then with its help the unknown polynomial $z = \omega_m(\zeta)$ is constructed. This circumstance also appears to be convenient enough at solving practical problems by method of conformal mapping.

The function $z = \omega(\zeta)$, conformally mapping the unit disk G to the domain D, is analytic in the disk G and continuous in $\overline{G}$, hence, it can be represented in this disk by its Taylor series:

$$\omega(\zeta) = \sum_{k=0}^{\infty} C_k \zeta^k, \quad C_k = C_k' + iC_k''. \tag{4.5}$$

Since the function $z = \omega(\zeta)$ is analytic in the disk G and continuous in $\overline{G}$, due to the Cauchy integral formula the coefficients of the series (4.5) are defined by formula [1]

$$C_k = \frac{\omega^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\omega(t)\, dt}{t^{k+1}} \quad (k = 0, 1, \dots). \tag{4.6}$$

By virtue of the principal of correspondence of boundaries at conformal mapping the series (4.5) is convergent on γ, and in G the series (4.5) converges uniformly. Evidently, via the convergence of the series (4.5), $C_k \to 0$ when $k \to \infty$. If now instead of the series (4.5) we take its piece consisting of $m + 1$ terms

$$\omega_m(\zeta) = \sum_{k=0}^{m} C_k \zeta^k,$$

the relation $z = \omega_m(e^{i\varphi})$ $(0 \leq \varphi \leq 2\pi)$ will map the circle $|\zeta| = 1$ generally not to the contour S, but to the some contour S_m. Since for the points $\zeta \in \overline{G}$, $\omega_m(\zeta) \to \omega(\zeta)$ for $m \to \infty$, hence $S_m \to S$ for $m \to \infty$.

Thus, if we take m sufficiently large, the deflection of the contour S_m from the given contour S can be made arbitrarily small. After finding the coefficients C_k $(k = 0, 1, \dots, m)$ the contours S_m must be drawn. If the contour S_m has multiple points or the deflection of the contour S_m from the given boundary S is not in admissible limits, then the number m, i.e. the polynomial order, must be increased.

Let $\zeta = f(z)$ be a function conformally mapping the domain D to the unit disk G under normalization conditions (4.1). On the basic of conformality of the mapping the function $f(z)$ has inverse function $z = \omega(\zeta)$, which is determined uniquely, maps conformally the disk G to the domain D under conditions $\omega(\zeta_0) = z_0$, $\omega(\zeta_1) = z_1$. Otherwise, the functions f and ω are inverse.

In formula (4.6) under the integral sign there are boundary values of the functions $z = \omega(\zeta)$ and $\zeta = f(z)$, hence at numerical integrating relating the corresponding integral some circumstances, important from the numerical point of view should be taken into account. Depending on the geometry of the contour S and normalization conditions (4.1), it is not excluded that the images t_j $(j = 1, 2, \dots, M)$ of the points z_j $(j = 1, 2, \dots, M)$ which are equidistant with respect to the parameter φ on boundaries S, will be situated not even non-uniformly, but may be strongly overcrowded at certain points [28]. Therefore, in order to prevent the stopping of the computer due to possible division by the machine zeros and to achieve high accuracy using the proper quadrature formulas it is reasonable in the integral (4.6) to pass from the contour γ to the contour S. Consequently, after construction of the function $\zeta = f_N(z)$ we will have a scheme

$$C_k^N = \frac{1}{2\pi i}\int\limits_S \frac{z f_N'(z)\,dz}{[f_N(z)]^{k+1}} = \frac{1}{2\pi i}\sum_{j=1}^{l}\int\limits_{\alpha_j}^{\beta_j} \frac{z^j(\varphi)\, f_N'(z^j(\varphi))\,(z^j(\varphi))'\,d\varphi}{[f_N(z^j(\varphi))]^{k+1}} \tag{4.7}$$

for approximate values of coefficients C_k, where $z = z^j(\varphi)$ $(\alpha_j \le \varphi \le \beta_j)$ are corresponding parameter equations of smooth curves S_j. The possibility of such passage is evident by virtue of analyticity of the function $f_N(z)$ in $\overline{D}$.

It is easy to show that if the function $f_N(z)$ is constructed within given $\varepsilon(\varepsilon > 0)$, i.e. $|f_N(z) - f(z)| < \varepsilon$ for $z \in S$, then $|C_k^N - C_k| = O(\varepsilon)$ and $C_k^N \to C_k$ when $N \to \infty$, since the sequence of the functions $f_N(z)$ converges uniformly on $\overline{D}$ to the function $f(z)$. Thus using the formula (4.7) coefficients of the series (4.5) are determined within ε.

5. Numerical Examples

In the examples considered below the calculations were carried out by the MATLAB system.

Example 1. Let the domain D be the interior of the ellipse $S: x = 5\cos\varphi,\ y = 3\sin\varphi$ $(0 \le \varphi \le 2\pi)$, i.e., $S = S_1, l = 1, \alpha_1 = 0, \beta_1 = 2\pi$. In the role of a boundary function $g(\tau)$ we took the function which has one break point $\tau_1 = (0,3)$. In particular, the function $g(\tau) = \arg(\tau - \tau_1),\ \tau \in S,\ \tau \ne \tau_1$ was taken. It is evident that $g^+(\tau_1) = \pi,\ g^-(\tau_1) = 2\pi$ and $\pi \le \arg(\tau - \tau_1) \le 2\pi$ for $\tau \in S,\ \tau \ne \tau_1$. In the considered case the exact solution to the problem A has the form $u(z) = \arg(z - \tau_1),\ z \in \overline{D},\ z \ne \tau_1$.

The approximate expression $z = \omega_m(\zeta)$ of function $z = \omega(\zeta)$ in the polynomial form under normalization conditions: $\omega(0) = 0$ and $\omega(1) = 5$ was constructed by the method described in the Section 4.

In the considered case in the formula (4.7): $l = 1,\ \alpha_1 = 0,\ \beta_1 = 2\pi, z_1(\varphi) \equiv z(\varphi) = 5\cos\varphi + i3\sin\varphi$. The function $\zeta = f_N(z)$ is the approximate expression of the function $\zeta = f(z)$, which conformally maps the domain D onto the unit disk G under normalization conditions: $f(0) = 0,\ f(5) = 1$, i.e., the functions $f_N(z)$ and $\omega_n(\zeta)$ are mutually inverse. We constructed the expression of the function $\zeta = f_N(z)$ by the scheme, which is given in the Point 4.1. For construction of the function $f_N(z)$ in the role of the auxiliary contour is taken ellipse $\widetilde{S}: x = 8\cos\varphi, y = 6\sin\varphi,\ 0 \le \varphi \le 2\pi$, and the collocation and auxiliary points lie uniformly with respect to the parameter φ on the contours S and $\widetilde{S}$, respectively. The number of these points is $N = 100$.

If we take into account that an ellipse has two symmetry axes, then on the basis of the normalization conditions it is easy to show that all coefficients C_k of the Taylor series of the function $z = \omega(\zeta)$ are real numbers and all C_k

with even indices are equal to zero, i.e., $C_k'' = 0$, $C_{2k} = 0$ $(k = 0, 1, 2, \dots)$. The indicated circumstance holds also in calculations of the coefficients C_k^N.

Table 1(a)

$m = 200,\ M = 100000,\ \varepsilon = 0.1E-14,\ \delta = 0.6E-12$			
k	C_k^N	k	C_k^N
1	$0.35416244259515E+01$	51	$0.62089961643730E-04$
3	$0.74857202098445E+00$	53	$0.46727137562580E-04$
5	$0.30777021024151E+00$	55	$0.35237714377524E-04$
7	$0.15632045480507E+00$	57	$0.26624070955671E-04$
9	$0.88375528404711E-01$	59	$0.20151768391927E-04$
11	$0.53334217153624E-01$	61	$0.15278234402572E-04$
13	$0.33638961448325E-01$	63	$0.11601353218422E-04$
15	$0.21903947998325E-01$	65	$0.88222226634093E-05$
17	$0.14611110326638E-01$	67	$0.67180526649451E-05$
19	$0.99325641336952E-02$	69	$0.51223618612366E-05$
21	$0.68558062781214E-02$	71	$0.39104517355128E-05$
23	$0.47918518921518E-02$	73	$0.29887142553037E-05$
25	$0.33846323594660E-02$	75	$0.22867355403439E-05$
27	$0.24121175552648E-02$	77	$0.17514467786400E-05$
29	$0.17323021268172E-02$	79	$0.13427793408844E-05$
31	$0.12524269250369E-02$	81	$0.10304288161657E-05$
33	$0.91081478031834E-03$	83	$0.79143897874848E-06$
35	$0.66583008576670E-03$	85	$0.60839384158356E-06$
37	$0.48899810068219E-03$	87	$0.46806180958930E-06$
39	$0.36062357822837E-03$	89	$0.36037682962621E-06$
41	$0.26694855419727E-03$	91	$0.27767132565861E-06$
43	$0.19827851959047E-03$	93	$0.21409767374594E-06$
45	$0.14772938795337E-03$	95	$0.16519111775885E-06$
47	$0.11037904594961E-03$	97	$0.12753904441326E-06$
49	$0.82686450879585E-04$	99	$0.98530334380181E-07$

In the Tables 1(a) and 1(b): k is the number of coefficient C_k^N; m is an order of polynomial $z = \omega_m(\zeta)$; M is a number of quadrature nodes of Simpson's formula for calculation the integrals of type (4.7); ε and δ are *a posteriori* error estimates of construction of functions $\zeta = f_N(z)$ and $z = \omega_n(\zeta)$ respectively:

$$\varepsilon = \max_{z \in S} \big|\, |f_N(z)| - |f(z)| \big| \approx \max \big| |f_N(z_k)| - 1 \big|,$$

$$\delta = \max_{\tau \in \gamma} \big| \omega(\tau) - \omega_n(\tau) \big| \approx \max \big| z_k - \omega_n(\zeta_k) \big|,$$

where $\zeta_k = \frac{f_N(z_k)}{|f_N(z_k)|} \in \gamma$ (see (4.4)), and the points z_k $(k = 1, 2, \dots, M)$ are situated uniformly with respect to the parameter φ on the contours S. Evidently, when the number M is large enough, the values ε and δ practically represent the deflections of contours $\zeta = f_N(z)$ $(z \in S)$ from γ and S_n $(z = \omega_n(\tau))$ from S respectively. In numerical experiments we took $M = 10^5$. In tables, in order to be short, corresponding values are written with exponent E, i.e., E plays the role of base 10.

Table 1(b)

k	C_k^N	k	C_k^N
101	$0.76165166813413E-07$	151	$0.14085239638787E-09$
103	$0.58910457759183E-07$	153	$0.10997871519702E-09$
105	$0.45589884556607E-07$	155	$0.85889270414382E-10$
107	$0.35300075996333E-07$	157	$0.67093365235434E-10$
109	$0.27346735901415E-07$	159	$0.52423270595625E-10$
111	$0.21195803489319E-07$	161	$0.40970761909046E-10$
113	$0.16436205965822E-07$	163	$0.32023819384423E-10$
115	$0.12751259834320E-07$	165	$0.25042514373297E-10$
117	$0.98968627950787E-08$	167	$0.19580950983607E-10$
119	$0.76847311737123E-08$	169	$0.15321228687230E-10$
121	$0.59695336374258E-08$	171	$0.11989044761407E-10$
123	$0.46390287495779E-08$	173	$0.93747809485715E-11$
125	$0.36064691979387E-08$	175	$0.73475284019982E-11$
127	$0.28047978251925E-08$	177	$0.57469080279778E-11$
129	$0.21821174363268E-08$	179	$0.45065382403564E-11$
131	$0.16982817134966E-08$	181	$0.35290207511807E-11$
133	$0.13221805854435E-08$	183	$0.27687632853102E-11$
135	$0.10297034424411E-08$	185	$0.21659547681333E-11$
137	$0.80219172143181E-09$	187	$0.17058351279668E-11$
139	$0.62513859558244E-09$	189	$0.13392141899142E-11$
141	$0.48731157476241E-09$	191	$0.10510085024507E-11$
143	$0.37998096409364E-09$	193	$0.82263085380655E-12$
145	$0.29637368184362E-09$	195	$0.64766790153215E-12$
147	$0.23122300273274E-09$	197	$0.51363294358472E-12$
149	$0.18044667458472E-09$	199	$0.40420744825317E-12$

If we take pieces of the series with 100 and 150 first terms, we will have $\delta = 0.1E-05$ and $\delta = 0.1E-08$, respectively.

After construction of the function $z = \omega_m(\zeta)$ the solution of the Problem B for the boundary function $g^*(t) \equiv g(\omega_m(e^{i\varphi})) = \arg(\omega_m(e^{i\varphi}) - \tau_1)$ we can calculate Poisson's integral (3.4) in the following form (in order an indeterminacy of type $\arg(0)$)

$$\int_0^{2\pi} = \int_0^{\pi/2-\varepsilon} + \int_{\pi/2+\varepsilon}^{2\pi} .$$

The values of approximate solution of the Problem B for the various points $\zeta_k \in G$ and $\varepsilon = 10^{-7}$ and the values of the exact solution to the problem A at the points $z_k = \omega_m(\zeta_k)$ are given in the Tables 2(a), 2(b).

Theoretically, must be $u^*(\zeta_k) = u(z_k)$, that takes place with high precision.

Table 2(a)

k	ζ_k	$u^*(\zeta_k)$
1	$(0,0)$	$3\pi/2$
2	$(0.5,0.5)$	5.67362865562897
3	$(-0.5,0.5)$	3.75114930515210
4	$(-0.5,-0.5)$	4.40553613355578
5	$(0.5,-0.5)$	5.01924182726673
6	$(0.9999,0)$	5.74266312095585
7	$(0,0.9999)$	4.71238896838075
8	$(-0.9999,0)$	3.68211481648013
9	$(0,-0.9999)$	4.71238896889993

Table 2(b)

k	z_k	$u(z_k)$
1	$(0,0)$	$3\pi/2$
2	$(1.55673600185879, 1.91299484566255)$	5.67362865561925
3	$(-1.55673600185879, 1.91299484566255)$	3.75114930515013
4	$(-1.55673600185879, -1.91299484566255)$	4.40553613350178
5	$(1.55673600185879, -1.91299484566255)$	5.01924182726760
6	$(4.99883658720709, 0)$	5.74266313527330
7	$(0, 2.99978002624038)$	4.71238898038469
8	$(-4.99883658720709, 0)$	3.68211482549608
9	$(0, -2.99978002624038)$	4.71238898038469

Example 2. The domain D is the interior of the ellipse $S : x = 5\cos\varphi$, $y = 3\sin\varphi$, $0 \le \varphi \le 2\pi$, and in the role of $g(\tau)$ we took a function with four break points: $\tau_1 = \frac{1}{\sqrt{2}}(5,3), \tau_2 = \frac{1}{\sqrt{2}}(-5,3), \tau_3 = \frac{1}{\sqrt{2}}(-5,-3), \tau_4 = \frac{1}{\sqrt{2}}(5,-3)$. In particular, we took the function

$$g(\tau) = \begin{cases} x+y & \text{for} \quad \tau \in \tau_1\tau_2, \\ x+2 & \text{for} \quad \tau \in \tau_2\tau_3, \\ x-y & \text{for} \quad \tau \in \tau_3\tau_4, \\ x-2 & \text{for} \quad \tau \in \tau_4\tau_1, \end{cases}$$

where $\tau_1\tau_2,\ \tau_2\tau_3,\ \tau_3\tau_4,\ \tau_4\tau_1$ are open arcs of the contour S. For the points $\tau_k (k = 1,2,3,4)$ and the function $g(\tau)$ we have: $\arg\tau_1 = \pi/4,\ \arg\tau_2 = 3\pi/4,\ \arg\tau_3 = 5\pi/4,\ \arg\tau_4 = 7\pi/4; g^+(\tau_1) = 4\sqrt{2},\ g^-(\tau_1) = (5\sqrt{2} - 4)/2,\ g^+(\tau_2) = (4 - 5\sqrt{2})/2,\ g^-(\tau_2) = -\sqrt{2},\ g^+(\tau_3) = -\sqrt{2},\ g^-(\tau_3) = (4 - 5\sqrt{2})/2,\ g^+(\tau_4) = (5\sqrt{2} - 4)/2,\ g^-(\tau_4) = 4\sqrt{2}$.

Under the conformal mapping $\zeta = f_N(z)$ the points τ_k $(k = 1,2,3,4)$, respectively pass to the points $t_k \in \gamma$ (i.e., $t_k \leftrightarrow \tau_k$): $t_1 = (\cos\alpha, \sin\alpha)$, $t_2 = (-\cos\alpha, \sin\alpha)$, $t_3 = (-\cos\alpha, -\sin\alpha)$, $t_4 = (\cos\alpha, -\sin\alpha)$, where $\alpha = 0.324850059711$.

The values of approximate solution $u^*(\zeta)$ of the Problem B at the various points $\zeta_k \in G$ (or the values of approximate solution $u(z)$ to the Problem A at the points $z_k = \omega_n(\zeta_k)$) are given in the Table 3.

Table 3

k	ζ_k	$u^*(\zeta_k)$	z_k
1	$(0,0)$	2.20596975469616	$(0,0)$
2	$(0.5,0.5)$	3.75637943502893	$(1.55673600185879, 1.91299484566255)$
3	$(-0.5,0.5)$	1.01989634356777	$(-1.5567360185879, 1.91299484566255)$
4	$(-0.5,-0.5)$	1.01989634356777	$(-1.5567360185879, -1.91299484566255)$
5	$(0.5,-0.5)$	3.75637943502893	$(1.5567360185879, -1.91299484566255)$
6	$0.9999t_1$	3.59687209498660	$(3.53525689114854, 2.12085852856418)$
7	$0.9999t_2$	-1.47385215594198	$(-3.53525689114854, 2.12085852856418)$
8	$0.9999t_3$	-1.47385215594199	$(-3.53525689114854, -2.12085852856418)$
9	$0.9999t_4$	3.59687209499495	$(3.53525689114854, -2.12085852856418)$

Example 3. Let D be the exterior of the ellipse $S:\ x = 3\cos\varphi,\ y = \sin\varphi$ $(0 \le \varphi \le 2\pi)$. In the role of $g(\tau)$ we took the function with one break point $\tau_1 = (0,1)$. In particular, we took the function (see [18])

$$g(\tau) = \arg\left(i - \frac{\tau}{|\tau|^2}\right),\ \tau \ne \tau_1.$$

We have $0 \le g(\tau) \le \pi$ for $\tau \in S, \tau \ne \tau_1$ and $g^+(\tau_1) = 0, g^-(\tau_1) = \pi$. For the considered boundary function $g(\tau)$ the exact solution to Problem A has the form

$$u(z) = \arg(i - \frac{z}{|z|^2}),\ z \in \overline{D},\ z \ne \tau_1,$$

and it is evident that for the function $u(z)$ the condition $0 \le u(z) \le \pi,\ z \in \overline{D},\ z \ne \tau_1$ is fulfilled.

In the considered case the conformally mapping function has the form [22]

$$z = \omega(\zeta) = \frac{2}{\zeta} + \zeta,\ \ \zeta \in \overline{G}.$$

In the Table 4 the values of the approximate solution to the Problem B at various points $\zeta_k \in G$, and the exact solution to the problem A at the points $z_k = \omega(\zeta_k)$ are given. It is obvious from the Table 4 that the theoretical equality $u^*(\zeta_k) = u(z_k)$ is fulfilled with high precision.

Table 4

k	ζ_k	$u^*(\zeta_k)$	z_k	$u(z_k)$
1	$(0,0)$	$\pi/2$	∞	$\pi/2$
2	$(0.5,0.5)$	1.815774989993823	$(;2.5,-1.5)$	1.81577498992176
3	$(-0.5,0.5)$	1.32581766366275	$(-2.5,-1.5)$	1.32581766366803
4	$(-0.5,-0.5)$	1.22777238637277	$(-2.5,1.5)$	1.22777238637419
5	$(0.5,-0.5)$	1.91382026721442	$(2.5,1.5)$	1.91382026721560
6	$(0.9999,0)$	1.89253687462790	$(3.00010002000200,0)$	1.89253687949145
7	$(0,0.9999)$	1.57079632289806	$(0,-1.00030002000200)$	1.57079632679490
8	$(-0.9999,0)$	1.24905577086689	$(-3.00010002000200,0)$	1.24905577409834
9	$(0,-0.9999)$	1.57079632345903	$(0,1.00030002000200)$	1.57079632679490

From the Tables 2,3,4 it is seen that for the approximate solution to the Problem A in the neighborhood of the break points τ_k the condition of the Theorem 1 is fulfilled.

6. Concluding Remarks

The examples in the preceding section indicate the effectiveness of the proposed algorithm for approximate solving of Problem A. In particular, the algorithm is sufficiently simple for numerical realization and it is characterized by accuracy, which is practically sufficient for many problems.

From the Tables 2, 3, 4 it is seen that for the approximate solution the Problem A in the neighborhood of the break points τ_k the condition of the Theorem 1 is fulfilled.

Finally, it must be noted that the proposed algorithm we can apply for approximate solving such generalized Dirichlet three dimensional problems, which could be reduced to the problems of type A.

Acknowledgment

This work was supported by the grant GNSF/ST08/3-390.

References

1. M. A. Lavrent'jev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1973.
2. A. N. Tikhonov and A. A. Samarski, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1972.
3. V. S. Vladimirov, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1971.
4. G. A. Grinberg, The selected questions of mathematical theory of electric and magnetic phenomena. (Russian) *Izd. Akad. Nauk SSSR, Moscow*, 1948.
5. V. D. Kupradze and M. A. Aleksidze, The method of functional equations for the approximate solution of certain boundary value problems. (Russian) *Z. Vycisl. Mat. i Mat. Fiz.* **4** (1964), 683–715.
6. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili and T. V. Burchuladze, Three dimentional problems of the mathematical theory of elasticity and thermoelasticity. (Rassian) *Nauka, Moscow*, 1976.

7. S. Christiansen, On Kupradze's functional equations for plane harmonic problems. *Function Theoretic Methods in Differential Equations, Pitman Publishing, London*, 205–243, 1976.
8. R. Mathon and R.L. Johnston, The approximate solution of elliptic boundary-value problems by fundamental solutions. *SIAM J. Numer. Anal.* **14** (1977), No. 4, 638–650.
9. G. Fairweather and R. L. Johnston, The method of fundamental solutions for problems in potential theory. *Treatment of integral equations by numerical methods, Academic Press, London*, 349–359, 1982.
10. M. Katsurada and H. Okamoto, A mathematical study of the charge simulation method I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **35** (1988), No. 3, 507–518.
11. M. Katsurada, A matematical study of the charge simulation method. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989), No. 1, 135–162.
12. M. Katsurada, Asymptotic error analysis of the charge simulation method in a Jordan region with an analytic boundary. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **37** (1990), No. 3, 635–657.
13. T. Kitagawa, On the numerical stability of the method of fundamental solution applied to the Dirichlet problem. *Japan J. Appl. Math.* **5** (1988), No. 1, 123–133.
14. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems. Numerical treatment of boundary integral equations. *Adv. Comput. Math.* **9** (1998), No. 1–2, 69–95.
15. Y. S. Smyrlis and A. Karageorghis, Some aspects of the method of fundamental solutions for certain harmonic problems. *J. Sci. Comp.* **16** (2001), No. 3, 341–371.
16. L. V. Kantorovich and V. I. Krylov, Approximate methods of higher analysis. (Russian) 5-th edition, *Fizmatgiz, Moscow–Leningrad*, 1962.
17. M. A. Aleksidze, Fundamental functions in approximate solutions of boundary value problems. Part I. (Russian) *Nauka, Moscow*, 1991.
18. N. Koblishvili, Z. Tabagari and M. Zakradze, On reduction of the Dirichlet generaized boundary value problem to an ordinary problem for harmonic function. *Proc. A. Razmadze Math. Inst.* **132** (2003), 93–106.
19. A. Karageorghis, Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities. *Numer. Methods Partial Differential Equations* **8** (1992), No. 1, 1–19.
20. A. Poullikkas, A. Karageorghis and G. Georgiou, Methods of fundamental solutions for harmonic and biharmonic boundary value problems. *Comput. Mechanics* **21** (1998), No. 4–5, 416–423.
21. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems. *Adv. Comput. Math. Adv. Comput. Math.* **9** (1998), No. 1–2, 69–95.
22. N. I. Muskhelishvili, Some basic problems of mathematical theory of elasticity. (Russian) *Nauka, Moscow*, 1966.
23. P. F. Filchakow, Approximate methods of conformal mappings. (Russian) *Naukova Dumka, Kiev*, 1964.
24. T. K. DeLillo and A. R. Elcrat, A Fornberg-like conformal mapping method for slender regions. Computational complex analysis. *J. Comput. Appl. Math.* **46** (1993), No. 1–2, 49–64.
25. T. K. DeLillo, The accuracy of numerical conformal mapping methods: a survey of examples end rezults. *SIAM J. numer. Anal.* **31** (1994), No. 3, 788–812.
26. K. Amano, A biderectional method for numerical conformal mapping based on the charge simulation method. *J. Inform. Process.* **14** (1991), No. 4, 473–482.

27. Z. Natsvlishvili, Z. Sanikidze and M. Zakradze, On conformal mapping of a unit disk onto a finite domain. *Proc A. Razmadze Math. Inst.* **123** (2000), 93–103.
28. M. Zakradze, On some aspects of the solution of practical problems by the method of conformal mapping. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **16** (2001), No. 3, 111–117.
29. K. Amano, A charge simulation method for the numerical conformal mapping of in interior, exterior and doubly-connected domains. *J. Comput. Appl. Math.* **53** (1994), No. 3, 353–370.
30. Z. Natsvlishvili, Z. Sanikidze and M. Zakradze, On one approximate method of the conformal mapping. *Proc. A. Razmadze Math. Inst.* **120** (1999), 153–164.
31. L. Lazarashvili and M. Zakradze, On the method of fundamental solutions and some aspects of its application. *Proc. A. Razmadze Math. Inst.* **123** (2000), 41–51.

(Received 05.04.2012)

Authors' address:

Niko Muskhelishvili Institute of Computational
Mathematics of the Georgian Technical University
8a, Akuri Str., Tbilisi 0160
Georgia
E-mail: mamuliz@yahoo.com, z_sanikidze@yahoo.com