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ON REDUCTION OF THE DIRIGHLET GENERALIZED BOUNDARY VALUE PROBLEM TO AN ORDINARY PROBLEM FOR HARMONIC FUNCTION

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Abstract. The method of reduction of the Dirichlet generalized boundary value problem for a harmonic function to an ordinary problem is given in the case of finite multiply connected and infinite domains. The method is constructed on the basis of the scheme suggested by M. A. Lavrent'ev and B. V. Shabat, which can be applied only to a finite simply connected domain. Examples are considered and the results of numerical calculations are given.

რეზიუმე. ნაშრომში მოცემულია ჰარმონიული ფუნქციისათვის დირიხლეს განზოგადოებული სასაზღვრო ამოცანის ჩვეულებრივზე დაყვანის მეთოდი, სასრული მრავლადბმული და უსასრულო არის შემთხვევაში. აღნიშნული მეთოდი აგებულია მ. ვ. ლავრენტიევისა და ბ. ვ. შაბატის მეთოდის საფუძველზე, რომელიც გამოიყენება მხოლოდ სასრული ცალადბმული არისათვის. განხილულია მაგალითები და მოცემულია რიცხვითი ექსპერიმენტების შედეგები.

1. Introduction

Let a domain D in the plane $z = x + iy \equiv (x, y)$ be bounded by a piecewise smooth contour S without multiple points (i.e., S is a simple contour). Moreover, we assume that its parametric equation is given.

It is known that the classical statement of the Dirichlet ordinary boundary value problem for the Laplace equation requires the continuity of the boundary function. However, in practical problems there are cases when the boundary function is piecewise continuous and therefore it is necessary to consider the Dirichlet generalized problem (see [1,2]).

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A. On the boundary S of the domain D a function $g(\tau)$ is given which is continuous everywhere, except a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ at which it has discontinuities of the first kind. It is required to find a function $u(z) \equiv u(x,y) \in C^2(D) \bigcap C(\overline{D} \setminus \{\tau_1, \tau_2, \dots, \tau_n\})$ satisfying the conditions

$$\Delta u(z) = 0, \qquad z \in D \tag{1.1}$$

$$u(\tau) = g(\tau), \qquad \tau \in S, \quad \tau \neq \tau_k \quad (k = 1, \dots, n), \tag{1.2}$$

$$|u(z)| < M, \qquad z \in \overline{D}, \tag{1.3}$$

where Δ is The Laplace operator and M is a real constant.

It is known [1,2] that Problem (1.1)–(1.3) is correct and for the generalized solution $u(z)$ the generalized extremum principle is valid:

$$\min_{z \in S} u(z) < \underset{z \in D}{u(z)} < \max_{z \in S} u(z),$$

where for $z \in S$ it is assumed that $z \neq \tau_k$ $(k = \overline{1,n})$.

Note that the additional requirement of boundedness, when the domain D is finite, concerns actually only the neighborhoods of break points of the function $g(\tau)$. If the domain D is infinite, then condition (1.3) (except the above-mentioned) means that (see [3])

$$\lim u(z) = c, \quad \text{for} \quad z \to \infty, \tag{1.4}$$

where c is a real constant and $|c| < \infty$. Evidently, if the function $g(\tau)$ is continuous on S, then the Dirichlet generalized problem coincides with the ordinary problem.

Condition (1.3) plays an important role in the extremum principle (1.4) and, consequently, in the theorem on the uniqueness of a solution of Problem **A**. Indeed, for example, the function $u(z) = \mathrm{Re}(1 - 2/z)$ is harmonic in the disk $(x-1)^2 + y^2 < 1$ and equals zero everywhere on its boundary, except the point $z = 0$; nevertheless, inside the disk it is nonzero. Moreover, for unbounded boundary functions the theorem on the uniqueness of the solution of the Dirichlet generalized problem is invalid. For example, in the case of a disk $(x-1)^2 + y^2 < 1$ for $g(\tau) = 0$ as $\tau \neq 0$, there exist two harmonic functions $u(z) = \mathrm{Re}(1 - 2/z)$ and $u(z) = 0$ which take the given value.

It should be noted that the methods which are used for the solution of the Dirichlet ordinary boundary value problem are poorly suited (or not suited at all) for the solution of the Dirichlet generalized boundary value problem [4–9]. Therefore researchers try to conduct preliminary improvement of the posed boundary problem. More precisely, they try to reduce, if possible, Problem **A** by smoothing the boundary function $g(\tau)$ to the solution of the ordinary problem [4–5]. To this end, it is sufficient to have a function $u_0(z)$ which would be a solution of equation (1.1), bounded in $\overline{D}$, continuous in $\overline{D}$ everywhere, except the points $\tau = \tau_k$, and would have the same jumps at the points τ_k, as $g(\tau)$ has. Indeed, if such a function is constructed, then by introduction of a new unknown function

$$v(z) = u(z) - u_0(z), \tag{1.5}$$

for its determination we have already a Dirichlet ordinary problem

B.

$$\Delta v(z) = 0, \quad z \in D, \tag{1.6}$$

$$v(\tau) = f(\tau), \quad \tau \in S, \tag{1.7}$$

where $f(\tau)$ is a continuous function on the contour S (since the function $f(\tau)$ at the points τ_k would have removable discontinuities).

If the domain D is infinite, then for the uniqueness of the solution of Problems **B** and **A** (see [2,3]) we require additionally that

$$\lim v(z) = c_1, \quad \text{for} \quad z \to \infty, \tag{1.8}$$

$$\lim u_0(z) = c_2, \quad \text{for} \quad z \to \infty. \tag{1.9}$$

It is evident that in this case, since $c = c_1 + c_2$, c_2 must be given in advance and c_1 should be found while solving Problem (1.6),(1.7). Conditions (1.4), (1.8) and (1.9) are essential, respectively, for the uniqueness of the solution of Problems **A** and **B** in the case of an infinite domain. To see that this is so, it is sufficient, e.g., to consider an exterior of the disk with the center at the origin and of radius r, as the domain D, i.e., $S : |z| = r$. If the function $u_1(z)$ is a solution of Problem **A** without condition (1.4), then the functions of the type $u_2(z) = u_1(z) + k \ln \frac{|z|}{r}$ are the solutions of Problem **A**, where $k \neq 0$ is a real constant. It is clear that such a circumstance takes place also for Problem **B**. Thus without conditions (1.4), (1.8) and (1.9) the uniqueness of the solution of Problems **A** and **B** is violated.

After $v(z) = v(x, y)$ is constructed, (1.5) gives the needed solution $u(z)$,

$$u(z) = v(z) + u_0(z), \quad z \in \overline{D}, \quad z \neq \tau_k. \tag{1.10}$$

We consider all possible cases dealing with the domain D.

2. The Case of Finite Simply Connected Domain

In [1] a method is considered which allows one to reduce the solution of the Dirichlet generalized boundary value problem for finite simply connected domains to the solution of the ordinary problem. Here we give a brief account of the method. The function

$$u_0(z) = \sum_{k=1}^{n} u_k(z) \tag{2.1}$$

plays the role of the function $u_0(z)$, where

$$u_k(z) = \frac{h_k}{\delta_k} \arg(z - \tau_k). \tag{2.2}$$

In (2.2) h_k is the jump of the function $g(\tau)$ at the point τ_k, i.e., $h_k = g^+(\tau_k) - g^-(\tau_k)$, where $g^-(\tau_k)$ and $g^+(\tau_k)$ are the limit values of the boundary function $g(\tau)$, when τ tends to the point τ_k along S, respectively, in the positive and negative directions (by the positive direction is meant the movement along the boundary in the counter-clockwise direction); $\delta_k = \varphi_k^+ - \varphi_k^-$, where $\varphi_k^+ = \lim\limits_{\tau \to \tau_k +} \arg(\tau - \tau_k)$, $\varphi_k^- = \lim\limits_{\tau \to \tau_k -} \arg(\tau - \tau_k)$, $\tau \in S$; if τ_k is not an angular point, then $\delta_k = -\pi$; Here arg denotes the properly chosen branch of the argument. It is evident that the function $u_k(z)$ is harmonic in the domain D and continuous in $\overline{D}$ everywhere, except the points $\tau = \tau_k$. If $z \in D$ and $z \to \tau_k$ along the path, a tangent to which at the point τ_k makes an angle θ with the axis x (it can be easily seen that $\varphi_k^+ < \theta < \varphi_k^-$), then this function tends to the limit $\frac{h_k}{\delta_k}\,\theta$, i.e., the function $u_k(z)$ is bounded in $\overline{D}$. When passing along S in the positive direction through the point τ_k, the function $u_k(z)$ has a jump $\frac{h_k}{\delta_k}\,\varphi_k^+ - \frac{h_k}{\delta_k}\,\varphi_k^- = h_k$. If the function $u(z)$ is a solution of Problem A, then the function

$$v(z) = u(z) - \sum_{k=1}^{n} \frac{h_k}{\delta_k} \arg(z - \tau_k)$$

is harmonic in the domain D and continuous in $\overline{D}$. Indeed, $u(z)$ and all functions $u_k(z) = \frac{h_k}{\delta_k} \arg(z - \tau_k)$ are harmonic in D. Further, the limit values of the function $v(z)$ as $z \to \tau \neq \tau_k$ $(z \in D)$ are equal to

$$f(\tau) = g(\tau) - \sum_{k=1}^{n} u_k(\tau). \tag{2.3}$$

Moreover, the function $f(\tau)$ remains continuous while passing through each point τ_k. Indeed, it is seen from (2.3) that from the function $g(\tau)$ with a jump h_k at the point τ_k we subtract the function $u_k(\tau)$ with the same jump, and the rest of the terms of the sum (2.3) are continuous at that point. Thus the solution $u(z)$ of the Dirichlet generalized problem can be represented as the sum (1.10), where $v(z)$ is a solution of Problem B with the continuous boundary function (2.3) and the function $u_0(z)$ has form (2.1), i.e., finally we have

$$u(z) = v(z) + \sum_{k=1}^{n} \frac{h_k}{\delta_k} \arg(z - \tau_k). \tag{2.4}$$

It should be noted that since the ordinary Problem **B** is correct, i.e., the solution $v(z)$ exists, is unique and depends continuously on the data. Consequently, the generalized Problem **A** is correct or determined physically.

On the basis of formula (2.4) in [1] the following theorem is proved, which explains the behavior of the generalized solution in the neighborhood of the point τ_k.

Theorem 1. *When the point $z \in D$ approached the break point τ_k of the boundary function $g(\tau)$ along various paths, the solution $u(z)$ of the Dirichlet generalized problem can tend to any limit between $g^-(\tau_k)$ and $g^+(\tau_k)$.*

Remark 1. Under some restrictions, the case in which $n = 1$ has been considered in [4, p. 102-105]. Particularly, to reduce the Dirichlet generalized problem to the ordinary one it is assumed that at the break point τ_1 the tangent to S exits and the contour S near τ_1 lies on one side of the tangent.

3. The Case of an Infinite Plane with a Hole

Let a domain D be an infinite plane with the hole B_1 which is bounded by a simple contour S_1 ($S_1 \equiv S$). As we noted in Section 1, in this case the Dirichlet generalized boundary value problem has the form

P.

$$\Delta u(z) = 0, \quad \forall z \in D, \tag{3.1}$$

$$u(\tau) = g(\tau), \quad \tau \in S, \quad \tau \neq \tau_k \quad (k = \overline{1,n}) \tag{3.2}$$

$$|u(z)| < M, \quad z \in \overline{D}, \tag{3.3}$$

It can be easily shown that in this case the method of M. A. Lavrent'ev and B. V. Shabat (see Section 2) cannot be applied for reduction of Problem **P** to Problem **B**. Firstly, the requirement of continuity of the function $u_0(z)$ is violated in the domain D (see Section 2). Indeed, for example, under a single circuit around the contour S the function $\arg(z - z_k)$ increases by 2π (or -2π), i.e., the adequacy between real physical process and mathematical model violates. Moreover, the function $u_0(z)$ must have a finite limit at infinity. However, the last condition does not take place, since $\lim \arg(z-\tau_k)$ as $z \to \infty$ does not exist. To construct the desired function $u_0(z)$ for reduction of Problem (3.1)–(3.3) to Problem **B**, we pass from the infinite domain D to the finite simply connected domain D^* bounded by contour S^* ($S^* \equiv S_1^*$), using an elementary conformal mapping (inversion)

$$z^* = z_0 + \frac{R^2}{\overline{z} - \overline{z_0}}. \tag{3.4}$$

In (3.4), $\overline{z} = x - iy$, z_0 is the inner point of the domain B_1 (to avoid difficulties in calculations, it is better to take the "center" of B_1 as z_0) and R is a real constant which is the radius of the circumference G centered at the point z_0. It is obvious that it is not necessary for the domain B_1 to contain the circumference $G(z_0; R)$ entirely. In the case of the conformal mapping (3.4) the situation is as follows: the whole plane z is mapped onto the whole plane z^* and conversely. In particular, $z=\infty$ transforms into the point $z^*=z_0\in D^*$; the infinite domain D becomes the finite domain D^* and the contour S becomes the contour S^*.

From (3.4) we have

$$z=z_0+\frac{R^2}{\overline{z^*}-\overline{z_0}}. \tag{3.5}$$

It is easy to see that if mapping (3.4) defines the domain D^* with the boundary S^*, then with the help of mapping (3.5) Problem P transforms into Problem A for the domain D^*. Indeed, for the conformal mapping (3.5) the function $u(z)$ becomes the function $u^*(z^*)=u(z_0+\frac{R^2}{\overline{z^*}-\overline{z_0}})$ which is harmonic and continuous in $D^*\backslash\{z_0\}$ and bounded in the neighbourhood of the point z_0 (i.e., z_0 is a removable singular point). However, by virtue of the theorem on the elimination of singularities of a harmonic function (see [1–3]), the function $u^*(z^*)$ is harmonic in the domain D^*. Analogously, in the conformal mapping (3.5) the piecewise continuous function $g(\tau)$ $(\tau\in S)$ becomes the function $g^*(\tau^*)=g(z_0+\frac{R^2}{\overline{\tau^*}-\overline{z_0}})$ $(\tau^*\in S^*)$ which is piecewise continuous with conservation of the values of jumps. Thus, using mapping (3.5), we actually reduce Problem **P** to the problem

A*.

$$\Delta u^*(z^*)=0,\quad z^*\in D^*, \tag{3.6}$$

$$u^*(\tau^*)=g^*(\tau^*),\quad \tau^*\in S^*,\quad \tau^*\neq\tau_k^*, \tag{3.7}$$

$$|u^*(z^*)|<M,\quad z^*\in\overline{D^*}, \tag{3.8}$$

where $\tau_k^*=z_0+\frac{R^2}{\overline{\tau_k}-\overline{z_0}}$.

Since D^* is the finite simply connected domain, for reduction of Problem (3.6)–(3.8) to Problem B, we can use the method of M. A. Lavrent'ev and B. V. Shabat, i.e., we can use the function

$$u_0^*(z^*)=\sum_{k=1}^{n}u_k^*(z^*), \tag{3.9}$$

$$u_k^*(z^*)=\frac{h_k^*}{\delta_k^*}\arg(z^*-\tau_k^*), \tag{3.10}$$

where $h_k^*\equiv h_k=g^+(\tau_k)-g^-(\tau_k)$.

If we insert the values z^* and τ_k^* from (3.4) into (3.9) and (3.10) (as radius R is taken $R=1$), then we obtain the needed function $u_0(z)$ for an infinite domain D in the form

$$u_0(z)=\sum_{k=1}^{n}u_k(z), \tag{3.11}$$

$$u_k(z) = \frac{h_k}{\delta_k} w_k(z), \quad w_k(z) = \arg\left(\overline{\frac{z-\tau_k}{(z-z_0)(z_0-\tau_k)}}\right), \quad \delta_k = \varphi_k^+ - \varphi_k^-,$$
$$\varphi_k^+ = \lim_{\tau\to\tau_k+} w_k(\tau), \quad \varphi_k^- = \lim_{\tau\to\tau_k-} w_k(\tau), \ \ \tau \in S.$$

From (3.11) for the value of the constant c_2 (see (1.9)) we have

$$c_2 = \lim_{z\to\infty} u_0(z) = \sum_{k=1}^{n} \frac{h_k}{\delta_k} \arg(z_0 - \tau_k).$$

It should be noted that since in the case of the conformal mapping (3.5) Problems **P** and A^* transform into each other, for the generalized solution of Problem **P** there is a theorem which is analogous to Theorem 1.

4. The Case of a Finite Multiply Connected Domain

Let a domain D be the finite m-connected domain with the boundary $S = \bigcup_{k=1}^{m} S_k$, where each S_k is a closed simple contour. Moreover, $S_k \bigcap S_j = \emptyset$, when $k \neq j$ and the contours S_i $(i = 1, 2, \ldots, m-1)$ lie inside the finite domain which is bounded by the contour S_m. Note that the method which is described in Section 2 can be applied to this case, if all points τ_k $(k = 1, 2, \ldots, n)$ of discontinuity are placed on the contour S_m. If either $\tau_k \in S_j$ and $j \neq m$, then the continuity of the function $u_0(z)$ is violated in the domain D. For example, under a single circuit around the contour S_j the function $\arg(z - \tau_k)$ increases by 2π (or -2π).

On the basis of Section 3, to avoid this circumstance we propose the method for reduction of Problem A to Problem B. It is evident that in Problem A it is not necessary that the points of discontinuity were placed on all contours S_k $(k = 1, 2, \ldots, m)$. For simplicity of our writing we introduce the following notation. We denote by $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$, where $1 \leq l \leq m$, those of the contours S_k $(k = 1, 2, \ldots, m)$ on which the points of discontinuity lie, and suppose that the number of the points of discontinuity on the contour Γ_i is k_i. It is clear that for the natural numbers k_i we have $1 \leq k_i \leq n$ and $k_1 + k_2 + \cdots + k_l = n$. Further, we denote by τ_{ik} $(k = 1, 2, \ldots, k_i)$ the points of discontinuity which lie on the contour Γ_i and analogously to Section 1 we introduce the notation $h_{ik} = g^+(\tau_{ik}) - g^-(\tau_{ik})$, where h_{ik} is a jump of the function $g(\tau)$ at the point τ_{ik}.

For smoothing of the function $g(\tau)$ on the contour Γ_i we consider the function

$$u_i(z) = \sum_{k=1}^{k_i} u_{ik}(z),$$

$$u_{ik}(z)=\begin{cases}\dfrac{h_{ik}}{\delta_{ik}}\arg\Big(\overline{\dfrac{z-\tau_{ik}}{(z-z_{i0})(z_{i0}-\tau_{ik})}}\Big) & \text{for } \Gamma_i\neq S_m,\\ \dfrac{h_{mk}}{\delta_{mk}}\arg(z-\tau_{mk}) & \text{for } \Gamma_i=S_m,\end{cases}\tag{4.1}$$

where z_{i0} is a "center" of the finite domain B_i with the boundary Γ_i ($z_{i0}\in B_i$), while

$$\delta_{ik}=\varphi_{ik}^{+}-\varphi_{ik}^{-},\quad \varphi_{ik}^{+}=\lim_{\tau\to\tau_{ik}+}\arg\Big(\overline{\frac{\tau-\tau_{ik}}{(\tau-z_{i0})(z_{i0}-\tau_{ik})}}\Big),$$
$$\varphi_{ik}^{-}=\lim_{\tau\to\tau_{ik}-}\arg\Big(\overline{\frac{\tau-\tau_{ik}}{\tau-z_{i0})(z_{i0}-\tau_{ik})}}\Big),\quad \tau\in\Gamma_i,\quad \Gamma_i\neq S_m;$$
$$\delta_{mk}=\varphi_{mk}^{+}-\varphi_{mk}^{-},\quad \varphi_{mk}^{+}=\lim_{\tau\to\tau_{mk}+}\arg(\tau-\tau_{mk}),$$
$$\varphi_{mk}^{-}=\lim_{\tau\to\tau_{mk}-}\arg(\tau-\tau_{mk}),\ \ \tau\in S_m.$$

It is easy to see that the function $u_i(z)$ is harmonic in the domain D and continuous in $\overline{D}$ everywhere, except the points τ_{ik} ($k=1,2,\dots,k_i$). The continuity of the function $u_i(z)$ is not violated in the domain D under the circuit around the contour Γ_i and, when passing along the contour Γ_i in the positive direction through the points τ_{ik}, the function has a jump h_{ik}.

As the function $u_0(z)$ we take

$$u_0(z)=\sum_{i=1}^{l}u_i(z)=\sum_{i=1}^{l}\sum_{k=1}^{k_i}u_{ik}(z)$$

and consider the function

$$v(z)=u(z)-\sum_{i=1}^{l}\sum_{k=1}^{k_i}u_{ik}(z),$$

where $u(z)$ is a solution of Problem **A** for the domain D with the boundary function $g(\tau)$. Let us show that the function $v(z)$ is a solution of Problem **B**. Indeed, $u(z)$ and all the functions $u_{ik}(z)$ (see (4.1)) are continuous and harmonic in the domain D. Further, the limit value of the function $v(z)$, as $z\to\tau\neq\tau_{ik}$ ($z\in D$) is

$$f(\tau)=g(\tau)-\sum_{i=1}^{l}\sum_{k=1}^{k_i}u_{ik}(\tau).\tag{4.2}$$

The function $f(\tau)$ remains continuous while passing through every point τ_{ik}. Indeed, in (4.2), from the function $g(\tau)$ with a jump h_{ik} at the point τ_{ik} we subtract the function $u_{ik}(\tau)$ with the same jump and the rest terms of the sum (4.2) are continuous at this point.

Thus in the case of the finite multiply connected domain D the solution $u(z)$ of Problem **A** can be represented in the form

$$u(z) = v(z) + \sum_{i=1}^{l} \sum_{k=1}^{k_i} u_{ik}(z). \tag{4.3}$$

In (4.3) $v(z)$ is a solution of Problem **B** with the boundary function $f(\tau)$ and the functions $u_{ik}(z)$ are given by (4.1). Taking into account the unique solvability of Problem **A**, the unequeness of the obtained solution is clear. In the case under consideration, on the basis of (4.3), there is a theorem which is analogous to Theorem 1 and explains the behavior of the generalized solution in the neighborhood of the point τ_{ik}.

Theorem 2. *The limit values of the solution $u(z)$ of the Dirichlet generalized problem, when the point $z \in D$ approaches the point τ_{ik} lie between $g^-(\tau_{ik})$ and $g^+(\tau_{ik})$.*

Indeed, let $z \in D$ and $z \to \tau_{ik}$ along the path, a tangent to which at the point τ_{ik} makes an angle θ with the axis x. On the basis of Sections 2 and 3, it can be easily shown that for an arbitrary contour Γ_i $(1 \le i \le l)$ we have $\varphi_{ik}^+ < \theta < \varphi_{ik}^-$. Further, (4.3) implies that the function $u(z)$ tends to the limit

$$u_\theta(\tau_{ik}) = \tilde{u}(\tau_{ik}) + \frac{h_{ik}}{\delta_{ik}}\theta. \tag{4.4}$$

In (4.4) $\tilde{u}(\tau_{ik})$ is the limit value of the sum of the function $v(z)$ and all those functions (which are involved in (4.3)) whose limits are independent of the path of approach of the point z to the point τ_{ik}, except the function $u_{ik}(z)$. In particular, if the point z approaches the point τ_{ik} along the contour Γ_i in the positive direction, then from (4.3) we have

$$g^+(\tau_{ik}) = \tilde{u}(\tau_{ik}) + \frac{h_{ik}}{\delta_{ik}}\varphi_{ik}^+. \tag{4.5}$$

Analogously,

$$g^-(\tau_{ik}) = \tilde{u}(\tau_{ik}) + \frac{h_{ik}}{\delta_{ik}}\varphi_{ik}^-. \tag{4.6}$$

With the help of (4.5) and (4.6), from (4.4) we get

$$u_\theta(\tau_{ik}) = g^+(\tau_{ik}) + \frac{h_{ik}}{\delta_{ik}}(\theta - \varphi_{ik}^+), \tag{4.7}$$

$$u_\theta(\tau_{ik}) = g^-(\tau_{ik}) + \frac{h_{ik}}{\delta_{ik}}(\theta - \varphi_{ik}^-). \tag{4.8}$$

In formulas (4.7) and (4.8) we know that $\delta_{ik} < 0$, $\theta - \varphi_{ik}^+ > 0$, $\theta - \varphi_{ik}^- < 0$ and $h_{ik} = g^+(\tau_{ik}) - g^-(\tau_{ik})$. Therefore, if $h_{ik} > 0$, then it is evident that $g^-(\tau_{ik}) < u_\theta(\tau_{ik}) < g^+(\tau_{ik})$. Analogously, if $h_{ik} < 0$, then $g^+(\tau_{ik}) < u_\theta(\tau_{ik}) < g^-(\tau_{ik})$. Theorem is proved.

5. The Case of an Infinite Plane with Holes

Let a domain D be the infinite plane with the holes D_k $(k = 1, 2, \dots, m)$, which are bounded respectively by simple contours S_k, i.e., the entire boundary of the domain D is $S = \bigcup\limits_{k=1}^{m} S_k$. Analogously to Sections 3 and 4, in this case also the method of M. A. Lavrent'ev and B. V. Shabat cannot be applied for reduction of Problem **A** to Problem **B**. In the notation of Section 3, for smoothing the boundary function $g(\tau)$ on the contour Γ_i $(i = 1, 2, \dots, l)$ we consider the function

$$u_i(z) = \sum_{k=1}^{k_i} u_{ik}(z),$$

$$u_{ik}(z) = \frac{h_{ik}}{\delta_{ik}} \arg\Big(\overline{\frac{z-\tau_{ik}}{(z-z_{i0})(z_{i0}-\tau_{ik})}}\Big), \tag{5.1}$$

where z_{i0} is the "center" of the finite domain B_i (see Section 4). As $u_0(z)$ we take

$$u_0(z) = \sum_{i=1}^{l}\sum_{k=1}^{k_i} u_{ik}(z), \tag{5.2}$$

and consider the function $v(z) = u(z) - u_0(z)$, where $u(z)$ is the solution of Problem **A** with the boundary function $g(\tau)$. Analogously to Sections 3 and 4, it can be easily shown that $v(z)$ is the solution of Problem **B**. From (5.2) we have

$$c_2 = \lim_{z\to\infty} u_0(z) = \sum_{i=1}^{l}\sum_{k=1}^{k_i} \frac{h_{ik}}{\delta_{ik}} \arg(z_{i0} - \tau_{ik}).$$

Thus, in this case the solution of Problem **A** can be represented in the form

$$u(z) = v(z) + \sum_{i=1}^{l}\sum_{k=1}^{k_i} u_{ik}(z),$$

where $v(z)$ is the solution of Problem **B** with the boundary function of form (4.2), while the functions $u_{ik}(z)$ are given by formula (5.1). In the case under consideration there is a theorem which is similar to Theorems 1 and 2. Finally, we make one remark concerning the calculation of the values δ_{ik}.

Remark 2. In numerical realization of the above-considered methods it is necessary to take into account the following circumstances: a) Numerical realization of the these methods requires the calculation of a function of the type $\arg\psi(z) = \arg(\psi_1(x,y) + i\psi_2(x,y))$ (where ψ_1 and ψ_2 are real functions), which in its turn means the calculation of functions of the type $\arctan\frac{\psi_2}{\psi_1}$. Therefore, in calculation of $\frac{\psi_2}{\psi_1}$ overflow may occur (i.e., stoppage of a computer). This case will take place when $\psi_2 \nrightarrow 0$ and $\psi_1 \to 0$ (while the point z moves in the domain $\overline{D}$). In order to avoid these cases it is sufficient to calculate the function $\arctan\frac{\psi_2}{\psi_1}$ by using the following formula

$$\arctan\frac{\psi_2}{\psi_1}=\begin{cases}\arcsin\dfrac{\psi_2}{\sqrt{\psi_1^2+\psi_2^2}} & \text{as } \psi_1\geq 0,\\ -\arcsin\dfrac{\psi_2}{\sqrt{\psi_1^2+\psi_2^2}} & \text{as } \psi_1<0.\end{cases}$$

b) Moreover, in numerical realization of the proposed methods we cannot infinitely approach the point τ_{ik}. Indeed, in this case in calculation of functions of the type $\arg\psi(z)$, there occur indeterminacies of type $\frac{0}{0}$. The calculation of such indeterminacies in a computer is impossible at all or may give wrong results. In numerical realization, from our point of view, the following approach is more appropriate. We circumscribe around each point of discontinuity τ_{ik}, as a center, the circumference $C_{ik}:|z-\tau_{ik}|=\varepsilon$. Since the functions $g(\tau)$ and $\arg\psi(\tau)$ are continuous in the neighborhood of the points τ_{ik}, for arbitrarily small $\varepsilon_1>0$ there exists $\varepsilon(\varepsilon_1)$ such that the following conditions are fulfilled: 1) each circumference C_{ik} intersects the contour Γ_i only at two points τ_{ik}^+ and τ_{ik}^-, which are situated respectively to the right and to the left of the point τ_{ik}; 2) there will be no difficulties in calculation of $\arg\psi(\tau_{ik}^\pm)$. 3) inequalities $|g^\pm(\tau_{ik})-g(\tau_{ik}^\pm)|<\varepsilon_1$ and $|\varphi_{ik}^\pm-\arg\psi(\tau_{ik}^\pm)|<\varepsilon_1$ will be fulfilled. It is evident that in this case in the role of values $g^\pm(\tau_{ik})$ and $\varphi_{ik}^\pm$ we can take respectively $g(\tau_{ik}^\pm)$ and $\arg\psi(\tau_{ik}^\pm)$ with accuracy ε_1. If parametric equation of contour Γ_i is $z=z_i(t)$ and $\tau_{ik}=z_i(t_{ik})$, then we can take $\varepsilon>0$ so small that as the points τ_{ik}^+ and τ_{ik}^- we can take $\tau_{ik}^+=z_i(t_{ik}+\varepsilon)$ and $\tau_{ik}^-=z_i(t_{ik}-\varepsilon)$, and the conditions 1), 2), 3) will be fulfilled.

6. Numerical Examples of Smoothing of the Boundary Functions

In this section we present the results of the numerical experiments which are performed on the basis of the proposed methods. In Examples 1, 2 and 3 as the domain D we consider finite domains and the infinite z-plane with one hole, i.e., $m=1$ and $S=S_1=\Gamma_1$. For the sake of simplicity, in Examples 1, 2 and 3 we take as the function $g(\tau)$ the same function which has four points $\tau_k (k=1,2,3,4)$ of finite discontinuity. The points τ_k lie on the contour S preserving the order of succession under the positive circuit of S. In particular, we take the function

$$g(\tau)=\begin{cases}1 & \text{for } \tau\in\tau_1\tau_2,\\ 2 & \text{for } \tau\in\tau_2\tau_3,\\ 3 & \text{for } \tau\in\tau_3\tau_4,\\ 4 & \text{for } \tau\in\tau_4\tau_1,\end{cases}$$

where $\tau_1\tau_2$, $\tau_2\tau_3$, $\tau_3\tau_4$, $\tau_4\tau_1$ are open arcs of the contour S. It is evident that the jumps of the function $g(\tau)$ at the points $\tau_k (k = 1, 2, 3, 4)$ are equal: $h_1 = -3$; $h_2 = 1$; $h_3 = 1$; $h_4 = 1$. According to Remark 2, as the points τ_k^+ and τ_k^- we take $\tau_k^+ = z(t_k + \varepsilon)$, $\tau_k^- = z(\tau_k - \varepsilon)$ respectively, where $z = z(t)$ is the parametric equation of the contour S, and t_k is the value of parameter t for which $\tau_k = z(t_k)$. In the numerical experiments $\varepsilon = 10^{-7}$ was taken and calculations were performed in double precision.

Example 1. The domain D is the interior of the ellipse $S: \ x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$. Since the contour S is smooth, $\delta_k = -\pi \approx -3.141592653589793$. In Table 1 the results are given for $a = 5$, $b = 2$ and $\tau_1 = (5, 0)$, $\tau_2 = (0, 2)$, $\tau_3 = (-5, 0)$, $\tau_4 = (0, -2)$.

Table 1

k	δ_k	$f(\tau_k^+)$	$f(\tau_k^-)$
1	-3.14159240425	1.49999981566	1.49999975817
2	-3.14159261362	0.98447561099	0.98447557439
3	-3.14159240514	1.49999979033	1.49999978463
4	-3.14159261362	2.01552400029	2.01552396483

Example 2. The domain D is the interior of the astroid $S: \ x = 2\cos^3 t$, $y = 2\sin^3 t$, $0 \leq t \leq 2\pi$. Having in mind the definition of φ_k^+ and φ_k^-, it should be noted that if the point τ_k is a point of cusp, then theoretically $\delta_k = \varphi_k^+ - \varphi_k^- = 0$. It is evident that for the engineering (practical) problems $\delta_k \neq 0$, however really it is near zero. The calculations show that in the engineering problems the above-described methods can be applied to cuspidal point. In Table 2 we present the results for the case when the points $\tau_k (k = 1, 2.3, 4)$ are the points of a cusp, namely $\tau_1 = (2, 0)$, $\tau_2 = (0, 2)$, $\tau_3 = (-2, 0)$, $\tau_4 = (0, -2)$.

Table 2

k	δ_k	$10^{-8} f(\tau_k^+)$	$10^{-8} f(\tau_k^-)$
1	$-0.133439988836E-06$	0.483674282386	0.483674282386
2	$-0.129905311397E-06$	-0.349943676833	-0.349943676833
3	$-0.133439988304E-06$	-0.222619109110	-0.222619109110
4	$-0.129905311397E-06$	-0.578968783773	-0.578968783773

Example 3. As an infinite domain D we take the exterior of the ellipse $S: x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$. In Table 3 we represent the results

of calculations for $a = 5$, $b = 2$, and $\tau_1 = (5, 0)$, $\tau_2 = (0, 2)$, $\tau_3 = (-5, 0)$, $\tau_4 = (0, -2)$.

Table 3

k	δ_k	$f(\tau_k^+)$	$f(\tau_k^-)$
1	−3.14159282218	1.50000046373	1.50000044398
2	−3.14159219356	2.01552450865	2.01552470368
3	−3.14159282218	1.50000043827	1.50000046944
4	−3.14159219356	0.98447620403	0.98447639906

Example 4. D is the doubly-connected domain with the boundary $S = S_1 \cup S_2$, where the contour $S_1 (S_1 \equiv \Gamma_1)$ is the ellipse $S_1 : \; x = a \cos t$, $y = b \sin t$, while the contour $S_2 (S_2 \equiv \Gamma_2)$ is the circumference $S_2 : x = r \cos t$, $y = r \sin t$, $0 \leq t \leq 2\pi$. As the boundary function we take the following one:

$$g(\tau) = \begin{cases} g_1(\tau), & \tau \in S_1, \\ g_2(\tau), & \tau \in S_2. \end{cases} \tag{6.1}$$

In (6.1) the functions $g_1(\tau)$ and $g_2(\tau)$ on the open arcs of the contours S_1 and S_2 respectively have the form

$$g_1(\tau) = \begin{cases} 1 & \tau \in \tau_{11}\tau_{12}, \\ 2 & \tau \in \tau_{12}\tau_{13}, \\ 3 & \tau \in \tau_{13}\tau_{14}, \\ 4 & \tau \in \tau_{14}\tau_{11}; \end{cases} \qquad g_2(\tau) = \begin{cases} 1 & \tau \in \tau_{21}\tau_{22}, \\ 3 & \tau \in \tau_{22}\tau_{23}, \\ 5 & \tau \in \tau_{23}\tau_{24}, \\ 7 & \tau \in \tau_{24}\tau_{21}. \end{cases}$$

The jumps of the function $g(\tau)$ at the points of discontinuity τ_{ik} $(i = 1, 2;$ $k = 1, 2, 3, 4)$ equal: $h_{11} = -3$, $h_{12} = 1$, $h_{13} = 1$, $h_{14} = 1$, $h_{21} = -6$, $h_{22} = 2$, $h_{23} = 2$, $h_{24} = 2$. The results of the calculations for $a = 5$, $b = 2$, $r = 10$, $\tau_{11} = (5, 0)$, $\tau_{12} = (0, 2)$, $\tau_{13} = (-5, 0)$, $\tau_{14} = (0, -2)$, $\tau_{21} = (10, 0)$, $\tau_{22} = (0, 10)$, $\tau_{23} = (-10, 0)$, $\tau_{24} = (0, -10)$ are given in Table 4.

Table 4

k	δ_{1k}	$f(\tau_{1k}^+)$	$f(\tau_{1k}^-)$
1	−3.14159282218	−0.499999503318	−0.499999713212
2	−3.14159219356	0.518188102194	0.518188348548
3	−3.14159282218	−0.499999664595	−0.499999553194
4	−3.14159219356	−1.518187565080	−1.518187321872

k	δ_{2k}	$f(\tau_{2k}^{+})$	$f(\tau_{2k}^{-})$
1	-3.14159255475	1.00000060017	1.00000018330
2	-3.14159255411	1.59033484048	1.59033494669
3	-3.14159255475	1.00000029723	1.00000048624
4	-3.14159255411	0.40966583678	0.40966594299

Finally, it should be noted that the numerical realization of the above-considered schemes has shown the following circumstance. According to Remark 2, in calculation of expressions of type $\arg\psi(\tau_{ik}^{\pm})$ (see Sections 2, 3, 4, 5) the "minimal" value of ε is 10^{-7}, i.e., if we take $\varepsilon = 10^{-8}$, then the considered methods are unstable.

References

1. M. A. Lavrent'jev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1973.
2. A. N. Tikhonov and A. A. Samarski, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1972.
3. V. S. Vladimirov, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1971.
4. G. A. Grinberg, The selected questions of mathematical theory of electric and magnetic phenomena. (Russian) *Izd. Akad. Nauk SSSR, Moscow-Leningrad*, 1948.
5. L. V. Kantorovich and V. I. Krylov, Approximate methods of higher analysis. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow-Leningrad*, 1962.
6. M. A. Aleksidze, Fundamental functions in approximate solutions of the boundary value problems, I. (Russian) *Nauka, Moscow*, 1991.
7. A. Karageorghis, Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities. *Numer. Methods Partial Differential Equations* **8**(1992), 1–19.
8. A. Poullikkas, A. Karageorghis and G. Georgiou, Methods of fundamental solutions for harmonic and biharmonic boundary value problems. *Comput. Mech.* **21**(1998), 416–423.
9. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems. *Adv. Comput. Math.* **9**(1998), 69–95.

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